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Approaches to Form-Factors of Higher Spin Heisenberg Chains

by

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"She had a pretty gift for quotation, which is a serviceable substitute for wit."

W. Somerset Maugham, *The Creative Impulse*

Abstract

In this thesis we apply the vertex operator approach of Jimbo and Miwa to higher spin Heisenberg chains with the aim of computing the form-factors of these quantum integrable models. The work is motivated by the relation of the form-factors to the dynamical structure factors of the model - objects that are experimentally realisable - and potential for comparison with real-world results.

Using a one boson, one fermion free field realisation of $U_q(\widehat{sl}_2)$, in conjunction with a realisation for the fermionic contribution due to Shiraishi, we are able to give the formalism required to obtain explicit multiple integral expressions for the $2m$-particle form-factors of the antiferromagnetic spin-1 Heisenberg chain. Using this novel boson-fermion-Shiraishi scheme, we are able to obtain single integral expressions for the two-spinon contribution to the S^+ form-factor.

We also consider a certain modification of a known q-Wakimoto bosonisation scheme for arbitrary spin and its relevance to the computation of higher spin form-factors. We consider the form of the resultant BRST relations and discuss simplifications arising through this approach, as well as the difficulties faced in obtaining integral expressions.

Acknowledgements

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I am also grateful (I think) to Jean-Sébastien Caux for suggesting that Robert and I 'throw ourselves off the form-factor cliff' when discussions began around the project that became this thesis and to Hitoshi Konno for both interesting discussions regarding the q-Wakimoto bosonisation and his patience with my many queries.

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Contents

Chapter 1

Introduction

The anisotropic spin-$\frac{1}{2}$ Heisenberg model is an interacting, one-dimensional quantum integrable system with quantum Hamiltonian

$$H_{XXZ} = -J\sum_{k=1}^{N}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \triangle(\sigma_k^z\sigma_{k+1}^z - 1)\right), \tag{1.1}$$

where σ_k^x, σ_k^y and σ_k^z are the Pauli spin-matrices acting on the kth site of the chain. If we set the anisotropy parameter Δ equal to 1, then we recover the XXX Heisenberg chain [1], which was studied using the first example of the Bethe ansatz in 1931 by Bethe [2]. From this starting point, the study of quantum integrable models in one-dimensional statistical mechanics really began. Several major advances were made in the area as the Bethe Ansatz approach was used to obtain solutions of the one-dimensional Bose gas by Lieb and Liniger [3], the six-vertex model by Lieb and Sutherland [4, 5], and the eight-vertex model by Baxter [6]. Within this period, a Bethe ansatz solution of the XXZ model was offered by Orbach in 1958 [7]. After many years of interest and progress in the area, including a more extensive offering from Baxter [8], an algebraic version of the Bethe ansatz was discovered by Faddeev, Sklyanin and Takhtajan [9–11], hinging on the underlying structure of the Yang-Baxter algebra.

The Yang-Baxter structures [8, 12–14] arising through the algebraic Bethe Ansatz [15] led to the development of the theory of quantum groups [16–20] and with this came the realisation that the quantum affine algebra $U_q(\widehat{sl}_2)$ is the underlying symmetry algebra of the Heisenberg spin chain in its infinite size limit [21, 22]. Out of this discovery came the vertex operator approach to solvable lattice models due to Jimbo and Miwa [23, 24], with the power of the approach really lying in Baxter's corner transfer matrix discovery [8]. In the vertex operator framework, correlation functions and form-factors are realised as traces of q-vertex operators ($U_q(\widehat{sl}_2)$ intertwiners) over irreducible highest weight $U_q(\widehat{sl}_2)$-modules. Using an appropriate free field realisation for $U_q(\widehat{sl}_2)$ then allows these objects to be explicitly computed in terms of traces of bosonic fields over Fock spaces.

Prior to the development of the vertex operator approach, key formulae for the use of the algebraic Bethe ansatz in the computation of correlation functions were put forward by Slavnov [25, 26]. Further progress in the area was then made, at the same time as Jimbo and Miwa developed their method, when a technique combining the solution of the quantum inverse problem with the algebraic Bethe ansatz was established. This involved the realization of local spin operators in terms of an operator called the monodromy matrix, which was to be defined on the entire chain [27, 28]. The first semi-numerical work on the use of the algebraic Bethe ansatz approach to correlation functions came early this century in a number of works [29–31] and progress has continued to be made ever since.

As a result of these explorations, we have access to two independent methods for treating the integrable spin chain and computing its correlation functions and form-factors: one based on the quantum group symmetry of the infinite chain and the other on the integrability of the finite lattice. There are advantages and disadvantages to both methods - the former can provide exact results, but is restricted to the zero field case and does not provide insight into finite size effects, whilst the latter can be applied to chains with non-zero magnetic field, but is restricted to finite size.

The beauty of choosing to consider the XXZ spin chain is that it is not only quantum integrable, allowing us to access this vast mine of techniques in order to compute correlation functions and form-factors, but it is also able to be physically realised in the lab due to the existence of so called quasi-one-dimensional materials. These magnetic materials are three-dimensional, but the interaction between their constituent parts is mainly along one-dimensional chains, whilst the interactions between the chains themselves is negligible. The one-dimensional spin chain lends itself perfectly to the modelling of such materials. Indeed, in the gapped antiferromagnetic regime ($J > 0$, $\Delta > 0$), the spin-$\frac{1}{2}$ chain (1.1) accurately describes quasi-one-dimensional materials $CsCoCl_3$ and $CsCoBr_3$ [32, 33], as discussed in [34]. The fully three-dimensional crystal $KCuF_3$ can also be realised as a one-dimensional antiferromagnet due to a structural distortion arising from the Jahn-Teller effect [35–38] described briefly in [39].

Objects of interest in the study of such materials in terms of their microscopic interactions are the so-called dynamical structure factors. These are expressed as the Fourier transforms of dynamical two-point correlation functions and, importantly, are realisable through experiments whereby neutrons or photons are scattered at the material of interest [40–42] . As we are able to use the techniques discussed above to compute such quantities theoretically, we have the exciting opportunity see the true power of the algebraic Bethe ansatz and quantum group approaches to integrable models in the comparison of their results with concrete experimental data.

In [43] and [44] the longitudinal structure factor $S^{zz}(k,\omega)$ is considered in the massless regime, where

$$S^{zz}(k,\omega) = \sum_{j\in\mathbb{Z}} e^{-ijk} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \mathrm{vac}| S^z_j(t) S^z_0(0) |\mathrm{vac}\rangle , \quad S^z = \frac{1}{2}\sigma^z,$$

in which t denotes time and $S^{zz}(k,\omega)$ is measurable in neutron scattering experiments where neutrons lose energy ω and momentum k. Computing such an object directly using either of the approaches discussed is not feasible as the correlation function is given by a raw expression involving j multiple integrals. However, inserting a complete set of spinon states[1] $\mathbb{I} = \sum_{\alpha} |\alpha\rangle \langle\alpha|$ allows the structure factor to be recast as

$$S^{zz}(k,\omega) = \sum_{\alpha} (2\pi)^2 \delta(k - K(\alpha))\delta(w - W(\alpha))| \langle \text{vac}| S_0^z |\alpha\rangle |^2,$$

where $W(\alpha)$ and $K(\alpha)$ are the energy and momentum of spinon state $|\alpha\rangle$. It is for this reason that we are primarily interested in computing form-factors of spin chains. The vertex operator approach was used in 1994 in order to compute spinon contributions to the dynamical correlation functions of the XXZ model [46]. The isotropic limit of this result was later used in order to compute the exact two-spinon structure factor of the spin-$\frac{1}{2}$ XXX chain, with these later results being compared effectively to finite-size results of the same model [47–49].

In recent years, successful progress has been made on the comparison of the spin-$\frac{1}{2}$ Bethe ansatz and algebraic vertex operator approach results for dynamical structure factors with one another, as well as their experimental counterparts [43, 44, 50–53]. In [52] finite size form-factors extracted from existing results in [27, 28, 54] and infinite size results obtained using the vertex operator approach [22, 23, 46] are used to compute the so called transverse dynamical structure factor of (1.1) in the gapped antiferromagnetic regime. In Fig. 5 of [52] the agreement between the infinite size results and finite size results for large N (N=1600) is demonstrated effectively. A nice review of some of the recent progress in the area, particularly in the finite size approach, is given in [39].

It is encouraging to see the amalgamation of these three different approaches - the finite size, infinite size and experimental - for obtaining results for dynamical structure factors of spin-$\frac{1}{2}$ Heisenberg chains. However, the scope of techniques available in the literature does not end there. In [34] we are challenged to 'roll up our sleeves and get to work' by using the wealth of literature available to compute the exact form-factors and correlation functions for models other than the spin-$\frac{1}{2}$

[1] Quantum soliton-like excitations in zero field are known as spinons [45].

chain, with a view to comparing them with Bethe ansatz results and real-world computations, analogously to [52].

Following discussions with Jean-Sébastian Caux and Rogier Vlijm regarding their plans to proceed with the algebraic Bethe ansatz computations for the dynamical structure factors of the antiferromagnetic spin-1 XXZ chain, it seemed apparent that using the vertex operator approach to compute explicit form-factors of higher spin chains was a good place to embark on this task. Further motivation for this choice of model comes from the fact that, like its spin-$\frac{1}{2}$ counterpart, the spin-1 antiferromagnet can be experimentally realised under certain conditions [55–58].

The aim of the project that eventually became this thesis was to explore the merits and limitations of existing bosonisation schemes, with the goal of computing exact results for, initially, the two-spinon contributions to spin-1 form-factors for local spin-1 operators S^+, S^- and S^z. The construction of the vertex operator approach exists for higher spin, as do appropriate free field realisations for $U_q(\widehat{sl}_2)$ [22, 59–67], however, a usable analytical expression[2] for spinon contributions to form-factors of the spin-1 XXZ chain does not appear to exist in the literature.

Whilst we have chosen to focus on the vertex operator approach, it should be noted that other recent work has applied different methods to the spin-1 model. In the finite temperature case, there are numerous works on the factorisation of spin-1 correlation functions [70, 71] and a similar approach has been applied in [72] for arbitrary temperature. The results of this work are exact algebraic expressions for correlation functions, although they are unfortunately limited to the case of the isotropic chain. In the recent work of Deguchi and Matsui multiple-integral formula for the zero-temperature correlation functions and form-factors of the XXZ chain in the *massless* case at finite size are obtained [73, 74] and the one-point functions are explicitly calculated in [75]. Were similar results available using their approach for the massive case in which we work, it would certainly be interesting to compare these results for the one-point function with our own method. More recently, Jimbo, Miwa and Smirnov put forward interesting work

[2]Usable is meant here in the sense of being able to directly extract numbers and make comparisons with Bethe ansatz results [68, 69] or experimental results.

on the construction of a 'fermionic basis' for the spin-1 XXZ model [76], with the aim of removing the need for multiple integrals in the results for correlation functions. The main outcome of that paper is described by the authors to be 'an existence theorem of a good basis in the space of quasi-local operators,' but it will be interesting to see what further results come from this approach.

Moving back to the vertex operator approach, Chapter 2 of the thesis contains background material outlining the treatment of the massive antiferromagnetic spin-$\frac{1}{2}$ XXZ chain (1.1), closely following the work of [22, 23]. We discuss the required representation theory of $U_q(\widehat{sl}_2)$ and also a free field realisation for the level one modules based on the Frenkel-Jing bosonisation [77]. With this framework established, in Chapter 3 we move to the spin-1 XXZ chain and consider a level two free field realisation for $U_q(\widehat{sl}_2)$ using one boson and one fermion, based on existing work in [50, 59–61].

In the form-factor expressions, we encounter difficulties when attempting to take the trace over the fermionic sector of our Fock space due to the appearance of objects called *fermion emission operators*. We resolve this using a method that we call the Shiraishi realisation for fermions. This is based on a free field realisation of the vertex operators of the Ising model hidden within a 2004 paper by Shiraishi [78] and its application in this setting is certainly novel. The theoretical framework for computing $2m$-particle form-factors of the massive spin-1 XXZ antiferromagnet is given in terms of this free field realisation. We are also able to analytically compute an explicit q-infinite product form for the two-point function of Ising vertex operators, previously conjectured in [22].

In Chapter 4, we are able to use the one-boson, one-fermion free field realisation along with the Shiraishi realisation for fermions in order to give explicit integral expressions for the two-spinon contributions to the S^+ form-factor, as we set out to achieve. The main results from this and the previous chapter will appear in [79]. Moving on to Chapter 5, we consider a different free field realisation for $U_q(\widehat{sl}_2)$ which has its advantages in being defined for arbitrary spin and level, meaning that it can be applied to higher spin Heisenberg chains. It also has the attractive property of not containing any fermions nor fermion emission operators.

We use a modification of the q-Wakimoto bosonisation considered in [62], [63] and [64], introducing a novel screening charge (arising through discussion with H. Konno) and BRST operator. Explicit computation of form-factors proves difficult due to the reducibility of the Fock space in which we embed our irreducible highest weight modules. This means that the traces of q-vertex operators are obtained as an infinite alternating sum of graded operators over graded Fock spaces. Chapter 6 concludes the thesis with a summary of the results obtained and discussion of their potential applications.

Chapter 2

Background

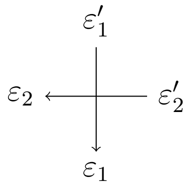
This chapter will introduce the objects and techniques required throughout the thesis. Following [23], we will consider how the vertex operator approach works in the setting of the spin-$\frac{1}{2}$ XXZ model before extending this to higher spin in the subsequent chapters. A nice introduction and review of related topics in quantum integrability is given in [80].

2.1 The Six-Vertex Model

The correlation functions and form-factors of the spin-$\frac{1}{2}$ XXZ quantum Hamiltonian (1.1) are equivalent to those of its associated two-dimensional statistical mechanical lattice model - the 6-vertex model. The vertex operator approach we would like to use arises more naturally in the diagrammatic setting of the lattice model and so this is what we will focus on to begin with.

We start by considering a two-dimensional square lattice with M vertical lines and N horizontal lines. On each edge with label j joining two vertices, we associate a spin variable ε_j, where $\varepsilon_j \in \{+, -\}$. The vertex model is a two-dimensional *classical* statistical mechanical model (as opposed to its associated spin-chain which is a one-dimensional *quantum* model), and as such the spin variables are ordinary commuting variables.

Around each vertex v we have one of six possible configurations and to each of these configurations, we assign a Boltzmann weight $R^{\varepsilon_1',\varepsilon_2'}_{\varepsilon_1,\varepsilon_2}$ as in Fig. 2.1.



$R^{\varepsilon_1',\varepsilon_2'}_{\varepsilon_1,\varepsilon_2}$.

We choose the Boltzmann weights

$$
\begin{aligned}
R^{++}_{++} &= R^{--}_{--} &&= \frac{1}{\kappa(z)}, \\
R^{+-}_{+-} &= R^{-+}_{-+} &&= \frac{1}{\kappa(z)}\frac{(1-z^2)q}{1-q^2z^2}, \\
R^{-+}_{+-} &= R^{+-}_{-+} &&= \frac{1}{\kappa(z)}\frac{(1-q^2)z}{1-q^2z^2}, \quad \kappa(z) = z\frac{(q^4z^2;q^4)_\infty(q^2z^{-2};q^4)_\infty}{(q^4z^{-2};q^4)_\infty(q^2z^2;q^4)_\infty},
\end{aligned}
$$

where we introduce the infinite-product notation $(x;p)_\infty = \prod_{n=0}^{\infty}(1-xp^n)$. We can associate the edges of the lattice with the two dimensional vector space $V = \mathbb{C}v_- \oplus \mathbb{C}_+$. The Boltzmann weight can then be interpreted as the factor we pick up when moving from $v_{\varepsilon_1'} \otimes v_{\varepsilon_2'}$ to $v_{\varepsilon_1} \otimes v_{\varepsilon_2}$, corresponding to the direction of our arrows. The particular choice of parametrisation is written in matrix form as

$$
R(z) = \frac{1}{\kappa(z)}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(1-z^2)q}{1-q^2z^2} & \frac{(1-q^2)z}{1-q^2z^2} & 0 \\ 0 & \frac{(1-q^2)z}{1-q^2z^2} & \frac{(1-z^2)q}{1-q^2z^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

where the structure is relative to the basis elements of $V \otimes V$ arranged in the order $v_+ \otimes v_+$, $v_+ \otimes v_-$, $v_- \otimes v_+$, $v_- \otimes v_-$. Thus, thinking of R as a matrix operator

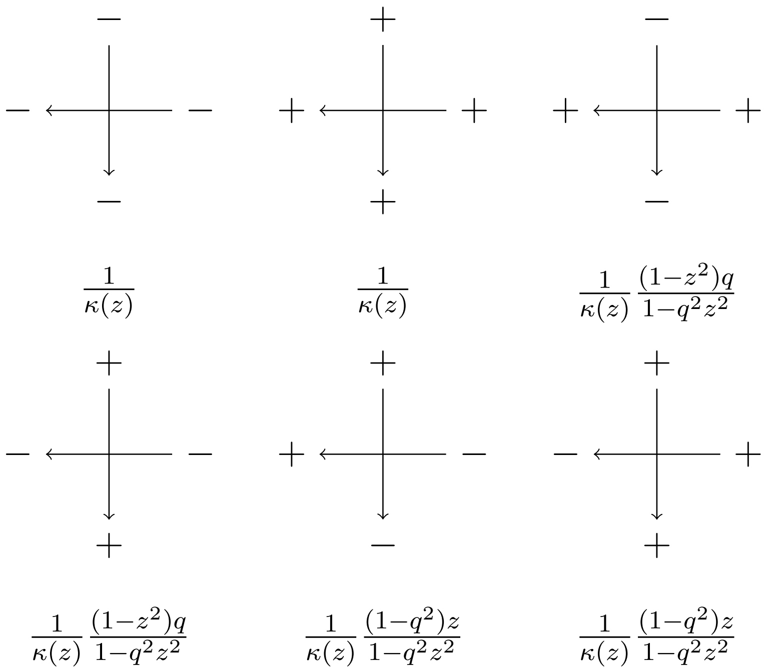


FIGURE 2.1: Configurations of the six-vertex model

$R \in \mathrm{End}(V \otimes V)$, we write

$$R(v_{\varepsilon'_1} \otimes v_{\varepsilon'_1}) = \sum_{\varepsilon_1,\varepsilon_2} v_{\varepsilon_1} \otimes v_{\varepsilon_2} R^{\varepsilon'_1,\varepsilon'_2}_{\varepsilon_1,\varepsilon_2}.$$

We think of q as being fixed and call z the spectral parameter, but really think of this as the ratio $z = z_1/z_2$, where z_1 and z_2 are spectral parameters attached to the first and second tensor components of $V \otimes V$ (or vertical and horizontal lattice spaces), respectively.

Given a lattice with a fixed configuration on each vertex, we say this lattice has configuration C. To each configuration C, we associate a configuration weight, $W(C)$. This is given by taking the product of the Boltzmann weights of all of the vertices of the configuration, i.e.

$$W(C) = \prod_v R^{\varepsilon'_1(v,C),\varepsilon'_2(v,C)}_{\varepsilon_1(v,C),\varepsilon_2(v,C)}, \tag{2.1}$$

where $\varepsilon_1(v, C)$ is the value of the spin on edge ε_1 of vertex v in configuration C. To find the probability of a particular configuration C taking place on the $M \times N$ square lattice, we normalise (2.1) by dividing by the partition function $Z_{M,N}$. This

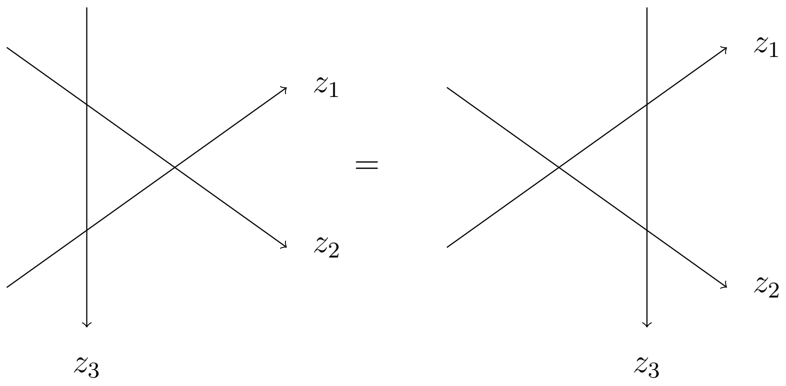


FIGURE 2.2: Yang-Baxter equation

is the sum of the weights of all possible configurations:

$$Z_{M,N} = \sum_{C} W(C). \tag{2.2}$$

2.1.1 Properties of the R-matrix

The particular choice of Boltzmann weights has its roots in the properties of the associated R-matrix $R(z)$. The most useful properties of the R-matrix for our purposes are listed below, along with their lattice interpretation, should it exist.

Yang-Baxter Equation

On the space $V_1 \otimes V_2 \otimes V_3$, where the V_k are copies of $V \equiv \mathbb{C}$, we have

$$R_{12}(z_1/z_2)R_{13}(z_1/z_3)R_{23}(z_2/z_3) = R_{23}(z_2/z_3)R_{13}(z_1/z_3)R_{12}(z_1/z_2), \tag{2.3}$$

where R_{ij} denotes the matrix R acting on spaces V_i and V_j as itself and as the identity on the remaining space. The variables z_k are spectral parameters associated with each V_k. The lattice interpretation is given in Fig. 2.2

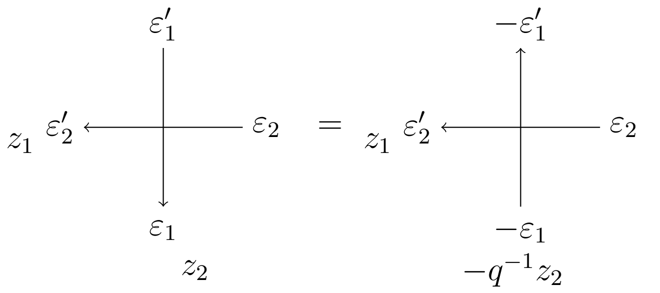


FIGURE 2.3: Crossing symmetry

Initial Condition

Defining P to be the permutation matrix, we have

$$R(1) = P.$$

Unitarity Relation

On $V_1 \otimes V_2$, we have

$$R_{12}(z_1/z_2)R_{21}(z_2/z_1) = 1. \tag{2.4}$$

Crossing Symmetry

We have

$$R(z_2/z_1)^{\varepsilon_2'\varepsilon_1}_{\varepsilon_2\varepsilon_1'} = R(-q^{-1}z_1/z_2)^{-\varepsilon_1'\varepsilon_2'}_{-\varepsilon_1\varepsilon_2}, \tag{2.5}$$

with lattice interpretation as shown in Fig. 2.3.

2.1.2 Equivalence with the XXZ Spin Chain

The transfer matrix of the XXZ model (1.1) with $\Delta = \frac{q+q^{-1}}{2}$, $|q| < 1$ is obtained by considering a single column of our six-vertex model lattice as shown in Fig.2.4. If we fix the spins $\varepsilon_N' \dots \varepsilon_1'$ and $\varepsilon_N \dots \varepsilon_1$ on the right and left (resp.) horizontal

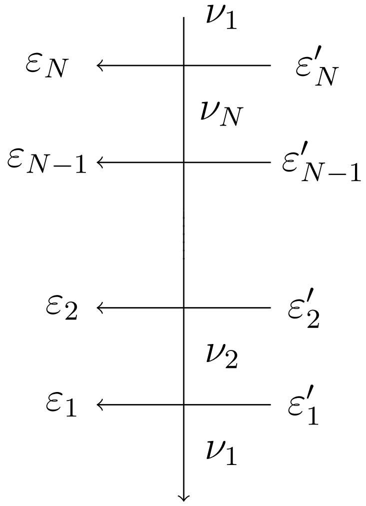


FIGURE 2.4: Transfer matrix

edges of this column and sum over the vertical edges $\nu_1, \nu_N, \ldots, \nu_1$ (imposing a periodicity condition) we obtain the object $T(z)$, which is a $2^N \times 2^N$ matrix with entries

$$T^{\varepsilon'_1 \ldots \varepsilon'_N}_{\varepsilon_1 \ldots \varepsilon_N} = \sum_{\nu_1, \ldots, \nu_N} R^{\nu_2 \varepsilon'_1}_{\nu_1, \varepsilon_1} R^{\nu_3 \varepsilon'_2}_{\nu_2, \varepsilon_2} \ldots R^{\nu_1 \varepsilon'_N}_{\nu_N, \varepsilon_N}, \tag{2.6}$$

acting on the tensor product $V^{\otimes N}$. We call $T(z)$ the transfer matrix. By definition, our partition function (2.2) is given by taking the trace of the product of M transfer matrices,

$$Z_{M,N} = \mathrm{Tr}\left(T^M(z)\right).$$

The XXZ Hamiltonian is obtained by taking the logarithmic derivative of the transfer matrix $T(z)$,

$$H_{XXZ} = \frac{1-q^2}{2q} z \frac{d}{dz} \log T(z) \Big|_{z=1}. \tag{2.7}$$

2.2 Quantum Affine Algebra $U_q(\widehat{sl}_2)$

As discussed in Chapter 1, the approach we will use hinges on the representation theory of quantum groups [20], introduced by Drinfeld and Jimbo [16, 17, 81].

With the lattice model introduced, we will now give the definition of the quantum affine algebra $U_q(\widehat{sl}_2)$, which is the underlying infinite-dimensional *non-Abelian* symmetry algebra of our model[1]. We use this terminology in contrast to the *Abelian* symmetries of the commuting transfer matrices which can be used in the context of the algebraic Bethe Ansatz method in order to 'solve' the model, see [11, 15, 80].

The algebra $U_q(\widehat{sl}_2)$ is a q-deformation of the universal enveloping algebra of the affine Lie algebra $\widehat{sl}_2$ and so we will start by fixing $\widehat{sl}_2$ notation. We have fundamental weights Λ_0, Λ_1 and null root δ. The span of these objects is called the weight lattice, P,

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta.$$

The simple roots of $\widehat{sl}_2$ are given by

$$\alpha_1 = 2\Lambda_1 - 2\Lambda_0, \quad \alpha_0 = \delta - \alpha_1,$$

and we also define an element ρ by $\rho = \Lambda_0 + \Lambda_1$. We introduce a symmetric bilinear form $(\ ,\ ) : P \times P \to \mathbb{Z}$ with relations

$$\begin{aligned}
&(\Lambda_0, \Lambda_0) = 0, \quad (\Lambda_0, \alpha_1) = 0, \quad (\Lambda_0, \delta) = 1 \\
&(\alpha_1, \alpha_1) = 2, \quad (\alpha_1, \delta) = 0, \quad (\delta, \delta) = 0.
\end{aligned}$$

We denote the dual lattice to P by P^*, where P^* has basis $\{h_0, h_1, d\}$, dual to $\{\Lambda_0, \Lambda_1, \delta\}$. The dual pairing is denoted as

$$\langle\ ,\ \rangle : P \times P^* \to \mathbb{Z},$$

[1] There is a useful table in [82] outlining the symmetries in various integrable models, including XXZ.

and by introducing the equality

$$\langle h_i, \lambda \rangle := (\alpha_i, \lambda), \quad \lambda \in P,$$

we are able to identify elements of P^* with elements of P as

$$h_0 = \alpha_0, \quad h_1 = \alpha_1, \quad d = \Lambda_0, \quad \rho = 2d + \frac{1}{2}h_1.$$

It should be noted that we are working with $|q| < 1$. With this set up, we are ready to introduce $U_q(\widehat{sl}_2)$.

Definition 2.1. *Quantum affine algebra* $U_q(\widehat{sl}_2)$

We define $U_q(\widehat{sl}_2)$ as an associative algebra over $\mathbb{C}$, with unit, generated by e_0, e_1, f_0, f_1 and q^h, $(h \in P^*)$. We have defining relations

$$\begin{aligned}
&q^h q^{h'} = q^{h+h'}, \quad q^0 = 1,\\
&q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i,\\
&q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i,\\
&[e_i, f_i] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}},\\
&e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0, \quad i \neq j,\\
&f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0, \quad i \neq j,
\end{aligned}$$

where $t_i = q^{h_i}$ and we use q-integer notation

$$[x] \quad = \quad \frac{q^x - q^{-x}}{q - q^{-1}}.$$

We equip $U_q(\widehat{sl}_2)$ with a Hopf algebra structure, introducing the coproduct Δ, antipode a and counit ε which act as follows:

$$\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,\\
&a(q^h) = q^{-h}, \quad a(e_i) = -t_i^{-1} e_i, \quad a(f_i) = -f_i t_i,\\
&\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0.
\end{aligned}$$

The axiomatic properties of Hopf algebra maps are listed in Appendix A for completeness.

Definition 2.2. *Quantum affine algebra* $U_q'(\widehat{sl}_2)$

The algebra $U_q'(\widehat{sl}_2)$ is defined as the subalgebra of $U_q(\widehat{sl}_2)$ generated by the restricted generators $\{e_i, f_i, t_i, (i = 0, 1)\}$.

2.3 Representations of $U_q(\widehat{sl}_2)$

In the constructions used in later sections, we require two types of representation of our quantum affine algebra - the evaluation modules and the irreducible highest weight modules (IHWM). We will also require the concept of tensor product and dual representations. We denote a representation on $U_q(\widehat{sl}_2)$-module V with action $\pi(x)$, $x \in U_q(\widehat{sl}_2)$ by (π, V).

2.3.1 Tensor Product Representations and Dual Modules

With the definition of the Hopf algebra structure, we can use the coproduct and the antipode in order to introduce tensor product and dual representations for $U_q(\widehat{sl}_2)$.

Definition 2.3. *Tensor product representations*

Given two representations, (π_V, V) and (π_W, W), of $U_q(\widehat{sl}_2)$, the tensor product representation is denoted $(\pi_{V\otimes W}, V \otimes W)$ and is given by

$$\pi_V \otimes \pi_W \quad = \quad (\pi_V \otimes \pi_W) \circ \Delta. \tag{2.8}$$

Definition 2.4. *Dual modules*

Given a representation (π, V) of $U_q(\widehat{sl}_2)$, the dual representation is denoted $(\pi^{*a^{\pm 1}}, V^*)$, where $V^* = \mathbb{C}v_+^* \oplus \mathbb{C}v_-^*$ is the (left) dual module with dual basis defined by

$\langle v_\varepsilon^*, v_{\varepsilon'}\rangle = \delta_{\varepsilon\varepsilon'}$. The action of $x \in U_q(\widehat{sl}_2)$ is given by

$$\langle x v_\varepsilon^*, v_{\varepsilon'}\rangle = \langle v_\varepsilon^*, a^{\pm 1} v_{\varepsilon'}\rangle. \tag{2.9}$$

2.3.2 Evaluation Modules

The algebra $U_q(\widehat{sl}_2)$ has a Hopf subalgebra $U_q(sl_2)$ which is generated by Chevalley generators e_1, f_1 and t_1. We can construct $U_q(\widehat{sl}_2)$ evaluation modules from finite-dimensional $U_q(sl_2)$ modules by introducing spectral parameters - sometimes called the *affinization* of the finite-dimensional module. For $k \in \mathbb{Z}_{>0}$, consider the $(k+1)$-dimensional vector space $V^{(k)}$ defined by

$$V^{(k)} = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \ldots \oplus \mathbb{C}v_k.$$

The pairing $(\pi, V^{(k)})$ gives a finite dimensional spin-$\frac{k}{2}$ representation of $U_q(sl_2)$ with action

$$\pi(t_1)v_j = q^{k-2j}v_j, \quad \pi(e_1)v_j = [j]v_{j-1}, \quad \pi(f_1)v_j = [k-j]v_{j+1}. \tag{2.10}$$

If we also define the action

$$\pi(t_0) = \pi(t_1)^{-1}, \quad \pi(e_0) = \pi(f_1), \quad \pi(f_0) = \pi(e_1),$$

in addition to relations (2.10), then we have the spin-$\frac{k}{2}$ representation of $U_q'(\widehat{sl}_2)$.

We now introduce z as our spectral parameter[2] and set $V_z^{(k)} = V^{(k)} \otimes \mathbb{C}[z, z^{-1}]$. Then, $(\pi_z, V_z^{(k)})$ is a $U_q(\widehat{sl}_2)$ representation with action

$$\begin{aligned}
&\pi_z(x) = \pi(x) \otimes 1, \quad x = e_1, f_1, t_1, t_0,\\
&\pi_z(e_0) = \pi(f_1) \otimes z, \quad \pi_z(f_0) = \pi(e_1) \otimes z^{-1}\\
&\pi_z(q^d)(v_j \otimes z^n) = q^n v_j \otimes z^n.
\end{aligned} \tag{2.11}$$

[2]In the literature, the symbol z is usually associated with the homogeneous gradation, whereas ζ would be associated with the principal gradation - we want to work in the homogeneous gradation as the level one modules we later use are adapted to this.

2.3.3 Irreducible Highest Weight Representations

We now set $P_+ = \mathbb{Z}_{\geq 0}\Lambda_0 + \mathbb{Z}_{\geq 0}\Lambda_1$. For any $\lambda \in P_+$, a $U_q(\widehat{sl}_2)$-module $V(\lambda)$ is called an irreducible highest weight module with highest weight λ if there exists a vector $|\lambda\rangle \in V(\lambda)$ such that [20, 83]

$$\begin{aligned} q^h \,|\lambda\rangle &= q^{(\lambda,\alpha)}\,|\lambda\rangle\,, \quad \alpha \in P \\ e_i \,|\lambda\rangle &= 0, \\ f_i^{(\lambda,\alpha_i)}\,|\lambda\rangle &= 0, \\ V(\lambda) &= U_q(\widehat{sl}_2)\,|\lambda\rangle\,. \end{aligned}$$

A $U_q(\widehat{sl}_2)$ module has level k if in that representation, $t_0 t_1 \simeq q^k$. Note that with this definition of the level of a representation, the evaluation modules have level 0 since $t_0 = t_1^{-1}$ in this case.

For a given $k \in \mathbb{Z}_{\geq 0}$, the level k irreducible highest weight modules are expressed in terms of the fundamental weights as $V(\lambda_\ell)$, $\ell = 0, 1, \ldots, k$ where $\lambda_\ell = (k - \ell)\Lambda_0 + \ell\Lambda_1$.

2.3.4 Drinfeld's Realisation of $U_q(\widehat{sl}_2)$

In [84], Drinfeld introduced a new realisation of $U_q'(\widehat{sl}_2)$ which lends itself more naturally to the free field realisation we will later consider. The notation in this section is consistent with that used in [23, 59–61]. In Chapter 5, we will alter this to be consistent with the appropriate existing literature and also to differentiate between bosonization schemes. The Drinfeld realisation of $U_q'(\hat{sl}_2)$ is generated by the letters $\{x_n^\pm | n \in \mathbb{Z}\}, \{a_n | n \in \mathbb{Z}_{\neq 0}\}, \gamma$ and K with relations

$$\begin{aligned}
[a_n, a_m] &= \delta_{m+n,0}\frac{1}{n}[2n]\frac{\gamma^n - \gamma^{-n}}{q - q^{-1}}, \\
[a_n, K] &= 0, \\
K x_n^\pm K^{-1} &= q^{\pm 2} x_n^\pm, \\
[a_n, x_m^\pm] &= \pm\frac{1}{n}[2n]\gamma^{\pm|n|/2} x_{n+m}^\pm, \\
x_{n+1}^\pm x_m^\pm - q^{\pm 2} x_m^\pm x_{n+1} &= q^{\pm 2} x_n^\pm x_{m+1}^\pm - x_{m+1}^\pm x_n^\pm, \\
[x_n^+, x_m^-] &= \frac{1}{q - q^{-1}}\left(\gamma^{(n-m)/2}\psi_{n+m} - \gamma^{(m-n)/2}\varphi_{n+m}\right), \\
\gamma &\in \text{the centre of the algebra,}
\end{aligned}$$

where $\{\psi_r, \varphi_s | r, s \in \mathbb{Z}\}$ are related to the generators $\{a_l | l \in \mathbb{Z}_{\neq 0}\}$ by the following.

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \psi_n z^{-n} &= K\exp\left\{(q - q^{-1})\sum_{k=1}^{\infty} a_k z^{-k}\right\}, \\
\sum_{n\in\mathbb{Z}} \varphi_n z^{-n} &= K^{-1}\exp\left\{-(q - q^{-1})\sum_{k=1}^{\infty} a_{-k} z^{k}\right\},
\end{aligned}$$

and $\psi_{-m} = \varphi_m = 0$ for $m > 0$. The bracket $[x, y]$ is the commutator $[x, y] = xy - yx$. The standard Chevalley generators of $U_q(\widehat{sl}_2)$, $\{e_i, f_i, t_i\}$, are given in terms of the Drinfeld generators by the identification

$$t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = x_0^+, \quad f_1 = x_0^-, \quad e_0 t_1 = x_1^-, \quad t_1^{-1} f_0 = x_{-1}^+. \tag{2.12}$$

2.3.5 Homogeneous and principal settings

The grading in the principal setting is defined by $D^{(i)} = -\rho + (\rho, \Lambda_i)$ and the character of the level one highest weight module $V(\Lambda_i)$, $i = 0, 1$ is given by

$$\begin{aligned}
\chi_{\Lambda_i}(x) &= Tr_{V(\Lambda_i)}\left(x^{D^{(i)}}\right) \\
&= \prod_{n=1}^{\infty} \frac{1}{1 - x^{2n-1}}.
\end{aligned} \tag{2.13}$$

In terms of the homogeneous grading operator, d, we have $\rho = 2d + \frac{\alpha}{2}$, where $\alpha = \alpha_1$.

2.4 The Vertex Operator Approach

Now that we have prepared the necessary tools, this section will introduce the vertex operator approach (VOA). This will involve a diagrammatic construction arising through consideration of the n-point correlation functions of the six-vertex model. Vertex operators and Baxter's corner transfer matrices arise naturally as lattice objects in this setting. The motivation behind this is our desire to compute the correlation functions and form-factors of the associated spin chain since, as previously mentioned, its correlation functions are equivalent to those of the six-vertex model. We will start by simply considering the one-point function. The construction that follows summarises the approach found in [23] and [82].

We start by considering a finite diamond-shaped lattice with a fixed ground state configuration on its boundary, as shown in Fig. 2.5[3]. The ith ground state configuration is a chequerboard type configuration with the spin on a reference edge labelled '0' set to $(-1)^{i+1}$, $i = 1, 0$. If we choose the ith ground state on the boundary, we say we are in the ith ground state sector. We do this with a view to taking the infinite limit and, as discussed in [23], we assume that the detailed shape of our lattice is unimportant as long as the edge that we will focus on is kept 'deep inside' when we pass to the infinite limit. We label the horizontal edges starting at the top by $N, N-1, \ldots, -N+2, -N+1$ and the vertical edges starting from the left by $N, N-1, \ldots, -N+1, -N$. To compute the one point function, we focus on the edge inside our lattice with label 0 and spin $\varepsilon_0 \in \{+,-\}$. The one point function ${}^{(i)}\langle \mathrm{vac}|\, \sigma_0^z \,|\mathrm{vac}\rangle^{(i)}$ is given by the sum

$$ {}^{(i)}\langle \mathrm{vac}|\, \sigma_0^z \,|\mathrm{vac}\rangle^{(i)} = P_+^{(i)} + P_-^{(i)}, \tag{2.14} $$

[3]The construction works in the same way with the other choice of ground state configuration $+ \leftrightarrow -$.

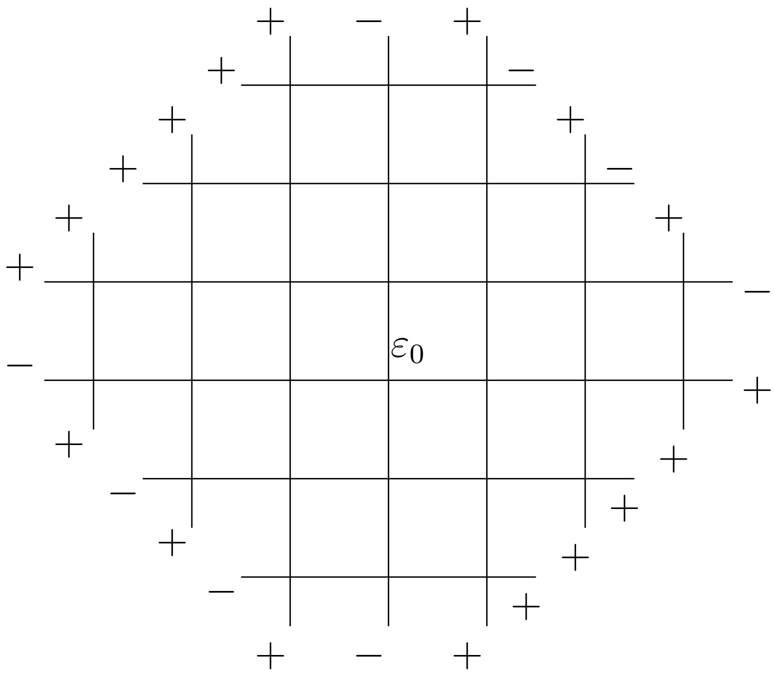


FIGURE 2.5: The one point function - finite diamond lattice

where $P_\varepsilon^{(i)}$ is the probability that the spin ε_0 on edge 0 takes the value ε in the ith ground state sector. Explicitly, in terms of our Boltzmann weights, these objects are given by the restricted sums

$$P_\varepsilon^{(i)} = \frac{\sum_{C'} \prod_v R_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1',\varepsilon_2'}(C',v)}{\sum_{C} \prod_v R_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1',\varepsilon_2'}(C,v)}, \tag{2.15}$$

where C' is restricted to the configurations in which $\varepsilon_0 = \varepsilon$ and the entire sum is restricted to configurations in our chosen ground state sector.

2.4.1 Baxter's Corner Transfer Matrix

To compute the sum $P_\varepsilon^{(i)}$ in terms of Boltzmann weights, we cut our specially shaped lattice into six sections - slicing along the dotted lines as shown in Fig. 2.6. The idea is then to sum over all of the internal spins, before summing over the spins on the seams. The two central columns are our lattice vertex operators and the superscript $(1-i,i)$ implies action from the ith to $(1-i)$th sectors. Zooming in on the upper half column in Fig. 2.7, we fix the spins $p_1,\ldots,p_N,p_1',\ldots,p_N'$ and

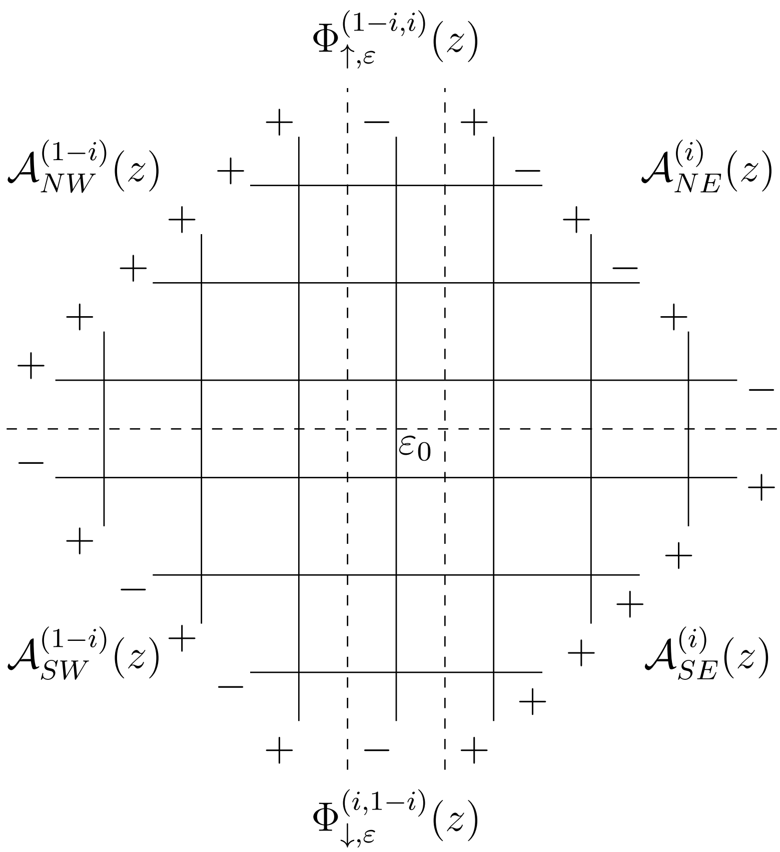


FIGURE 2.6: Splitting the finite lattice into six sections

sum the product of Boltzmann weights over the internal edges in order to define the entries of this $2^N \times 2^N$ half-column transfer matrix:

$$(\Phi_{\uparrow,\varepsilon})^{p'_N,\ldots,p'_1}_{p_N,\ldots,p_1} = \sum_{\nu_1,\ldots,\nu_{N-1}} \prod_{j=1}^{N} R^{\nu_j,p'_j}_{\nu_{j-1},p_j}, \tag{2.16}$$

where $\nu_0 = \varepsilon$ and $\nu_N = (-1)^{N+1-i}$, in correspondence with the chosen ground state sector. In a similar way, we can express the lower half column vertex operator in terms of Boltzmann weights as

$$(\Phi_{\downarrow,\varepsilon})^{p'_0,\ldots,p'_{-N+1}}_{p_0,\ldots,p_{-N+1}} = \sum_{\nu_{-1},\ldots,\nu_{-N+1}} \prod_{j=1}^{N} R^{\nu_j,p_j}_{\nu_{j-1},p'_j}. \tag{2.17}$$

The four corner regions are examples of *corner transfer matrices* introduced by Baxter [8]. The superscripts (i), again, refer to the ground state sectors on which they act. We can define them in a similar way to the vertex operators. Focusing

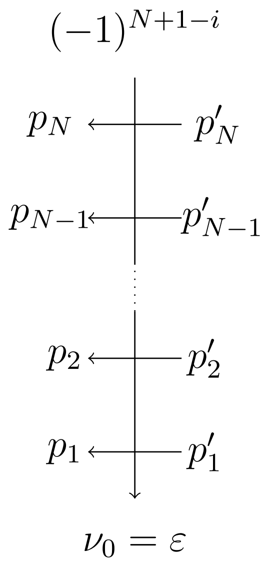


FIGURE 2.7: Lattice vertex operator $\Phi^{(1-i,i)}_{\uparrow,\varepsilon,}(z)$

this time on $\mathcal{A}^{(i)}_{NW}$, the north-west quadrant, we fix the horizontal boundary spins as $p_N, \ldots p_1$ and the vertical boundary spins as $p'_1, \ldots p'_N$. Entries of the $2^N \times 2^N$ corner transfer matrix are then given by

$$(\mathcal{A}_{NW})^{p'_N,\ldots,p'_1}_{p_N,\ldots,p_1} = \sum_E \prod_v R^{\varepsilon'_1(v),\varepsilon'_2(v)}_{\varepsilon_1(v),\varepsilon_2(v)}, \tag{2.18}$$

where the sum is taken over the set of all possible internal edges E and the product is taken over internal vertices, v. The other three corner transfer matrices are defined analogously.

It is now that the properties of the R matrix come into play, bringing some nice simplifications with them. If we choose our favourite lattice vertex operator $\Phi^{(1-i,i)}_{\varepsilon}(z) = \Phi^{(1-i,i)}_{\uparrow,\varepsilon}(z)$ and corner transfer matrix $\mathcal{A}^{(i)}(z) = \mathcal{A}^{(i)}_{NW}(z)$, we can use the R matrix crossing symmetry (2.5) to write all our other operators in terms of these two, only with shifted spectral parameters and some reversed spins. After playing this game, we end up with

$$\mathcal{A}^{(i)}_{SW}(z) = \mathcal{R}\mathcal{A}^{(i)}(-q^{-1}z^{-1}) \tag{2.19}$$

$$\mathcal{A}^{(i)}_{SE}(z) = \mathcal{R}\mathcal{A}^{(i)}(z)\mathcal{R} \tag{2.20}$$

$$\mathcal{A}^{(i)}_{NE}(z) = \mathcal{A}^{(i)}(-q^{-1}z^{-1})\mathcal{R} \tag{2.21}$$

$$\Phi^{(i,1-i)}_{\downarrow,\varepsilon,}(\xi) = \mathcal{R}\Phi^{(i,1-i)}_{-\varepsilon}(\xi)\mathcal{R}, \tag{2.22}$$

where $\mathcal{R}$ is the spin flip operator

$$\mathcal{R} : v_{\varepsilon_N} \otimes \ldots \otimes v_{\varepsilon_1} \mapsto v_{-\varepsilon_N} \otimes \ldots \otimes v_{-\varepsilon_1}.$$

This use of the R matrix crossing symmetry certainly helps to reduce the number of different operators we have to deal with, but the power of the construction we are considering really lies in Baxter's discovery regarding the infinite size limit of the corner transfer matrix. If the corner transfer matrix $\mathcal{A}^{(i)}(z)$ is normalised to make its largest eigenvalue equal to 1, then Baxter's discovery tell us that in the infinite lattice limit we have

$$\lim_{N\to\infty} A^{(i)} \propto z^{-D^{(i)}}, \tag{2.23}$$

where the operator $D^{(i)}$ is independent of z and has discrete integer spectrum $\{0, 1, 2, \ldots\}$[4]. We call this operator the corner transfer matrix Hamiltonian and it acts on the Hilbert space $\mathcal{H}^{(i)}$ spanned by its eigenvectors. We think of $\mathcal{H}^{(i)}$ as the limit of a certain subspace of the tensor product space $V^{\otimes N}$ with a span of half-infinite pure tensor vectors:

$$\ldots v_{k(3)} \otimes v_{k(2)} \otimes v_{k(1)}. \tag{2.24}$$

[4]Baxter's argument hinges on the Yang-Baxter equation (2.3) and the periodicity of the Boltzmann weights [8].

Towards the infinite limit on the left side, these basis vectors tend to an alternating ground state pattern, i.e.

$$k(j) = (-1)^{j+i}, \quad j \gg 0, \tag{2.25}$$

with the ground state in the ith sector corresponding to the half-infinite tensor product with $k(j) = (-1)^{j+i}$, $\forall j$.

The lattice vertex operator tends to some well defined operator in the same way as the corner transfer matrix, but (as noted when splitting up our original finite lattice), these carry one space to another because of the change in the ground state boundary configuration,

$$\Phi_{\varepsilon}^{(1-i,i)}(z) : \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(1-i)}.$$

Now, thinking back to our one-point function given in terms of $P_{\varepsilon}^{(i)}$, we were summing over all the spins inside our lattice and taking the product over all of the vertices. By definition, the corner transfer matrices and the lattice vertex operators sum over all the spins on their internal edges. What is now left to do is to take the sum over the spins on the seams between them. This amounts to taking the product of our matrices, moving around the lattice in an anticlockwise direction, and then taking the trace. Using the simplifications due to the crossing symmetry (2.19)-(2.20) and taking the limit (2.23), we have

$$\begin{aligned}
P_{\varepsilon}^{(i)} &= \mathcal{N}^{-1}\mathrm{Tr}\left(\mathcal{A}_{NE}^{(i)}(z)\mathcal{A}_{SE}^{(i)}(z)\Phi_{\downarrow,\varepsilon}^{(i,1-i)}(z)\mathcal{A}_{SW}^{(1-i)}(z)\mathcal{A}_{NW}^{(1-i)}(z)\Phi_{\uparrow,\varepsilon}^{(1-i,i)}\right) \\
&= \mathcal{N}^{-1}\mathrm{Tr}\left((-q)^{D^{(i)}}\Phi_{-\varepsilon}^{(i,1-i)}(z)(-q)^{D^{(1-i)}}\Phi_{\varepsilon}^{(1-i,i)}(z)\right) \\
&= \mathcal{N}^{-1}\mathrm{Tr}\left(q^{2D^{(i)}}\Phi_{-\varepsilon}^{(i,1-i)}(-q^{-1}z)\Phi_{\varepsilon}^{(1-i,i)}(z)\right),
\end{aligned}$$

where $\mathcal{N}$ is a normalisation factor ensuring that $P_+ + P_- = 1$. In the final step, we have used the homogeneity property of vertex operators

$$w^{-D^{(1-i)}} \circ \Phi_\varepsilon^{(1-i,i)}(z) \circ w^{D^{(i)}} = \Phi_\varepsilon^{(1-i,i)}(z/w),$$

which can be derived heuristically using the expression of the lattice vertex operator in terms of R matrices along with the Yang-Baxter equation [23]. All of this means that we now have our one point function given entirely in terms of lattice vertex operators and the corner transfer matrix Hamiltonian, D. The next step is to get something explicit out of these abstract expressions. For this, we look to the algebraic construction.

2.5 The Algebraic Construction

We consider, once again, the partition function of the six-vertex model. Computing the partition function in the setting of our diagrammatic construction amounts to taking the trace over our product of corner transfer matrices with no insertion of vertex operators. We associate the half infinite line on which our corner transfer matrices act with the Hilbert space $\mathcal{H}^{(i)}$ and, using (2.19)-(2.21) and (2.23), the partition function is proportional to the character

$$\mathrm{Tr}_{\mathcal{H}^{(i)}}\left(x^{D^{(i)}}\right) = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}},$$

where in our particular case, we set $x = q^2$. If we now think back to our consideration of the representation theory of $U_q(\widehat{sl}_2)$, we recall that the character of the $U_q(\widehat{sl}_2)$ highest weight module $V(\Lambda_i)$ is given by

$$\mathrm{Tr}_{V(\Lambda_i)}\left(x^{-\rho+i/2}\right) = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}},$$

where $\rho = 2d + \frac{\alpha}{2}$ in terms of the homogeneous grading operator. As discussed in [23], this motivates us to identify the highest weight module $V(\Lambda_i)$ with the space

$\mathcal{H}^{(i)}$ and the corner transfer matrix Hamiltonian $D^{(i)}$ with the grading operator of the algebra. The idea moving forward from here is to identify all of the key ingredients of our correlation functions (and also form-factors) with objects in representation theory. We will then have a solid algebraic setting in which to work. For more detail on the algebraic construction, the reader is directed to [22, 23, 82]. For now, we will state the results and identifications that will be used.

2.5.1 Dictionary between lattice and algebraic objects

Space of states

We regard the half-infinite tensor product of the upper column transfer matrix as representing the level one $U_q(\widehat{sl}_2)$ module $V(\Lambda_i)$:

$$\ldots \otimes V_z \otimes V_z \sim V(\Lambda_i),$$

where we specify boundary conditions of the ith ground state as the half infinite line extends towards infinity, as in (2.24), (2.25). To obtain the full infinite line, we need to consider the tensor product extending in both directions. If we slightly modify the antipode a introduced in Section 2.2 to give the anti-automorphism

$$b(x) = (-q)^{\rho} a(x)(-q)^{-\rho},$$

then we have the following isomorphism

$$\begin{aligned} G: \qquad V_z \quad &\simeq V_z^{*b} \\ v_{\pm} \otimes z^n \quad &\simeq \quad v_{\mp}^* \otimes z^n, \end{aligned}$$

and so we can write

$$\begin{aligned} V_z \otimes V_z \otimes \dots &\overset{G\otimes G\otimes\dots}{\simeq} V_z^{*b} \otimes V_z^{*b} \dots \\ &= (\dots \otimes V_z \otimes V_z)^{*b} \sim V(\Lambda_i)^{*b}. \end{aligned}$$

With this, we identify the infinite tensor product with the tensor product of the level one module tensored with its antipode dual with respect to b:

$$\dots V_z \otimes V_z \otimes V_z \otimes V_z \otimes \dots \sim V(\Lambda_i) \otimes V(\Lambda_j)^{*b},$$

where we have assumed we have the ith boundary condition to the left and the jth boundary condition to the right. Section 2 of [82] gives a detailed description of the embedding of the highest weight modules into the half infinite tensor product space. The full space of states of the infinite chain is then given by $\mathcal{F}$, where

$$\mathcal{F} = \mathcal{H} \otimes \mathcal{H}^{*b} = \oplus_{i,j=0,1} \mathcal{F}^{i,j} \tag{2.26}$$

$$\mathcal{H} = V(\Lambda_0) \oplus V(\Lambda_1), \tag{2.27}$$

$$\mathcal{F}^{i,j} = V(\Lambda_i) \otimes V(\Lambda_j)^{*b}. \tag{2.28}$$

Given a state $f \in \mathcal{F}$, we can regard this as a linear map on $\mathcal{H}$ by considering the relation $\mathcal{H} \otimes \mathcal{H}^{*b} \simeq \mathrm{End}(\mathcal{H})$. If we then take a linear operator $\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2$, where $\mathcal{O}_1 \in \mathrm{End}(\mathcal{H})$ and $\mathcal{O}_2 \in \mathrm{End}(\mathcal{H}^{*b})$, this will act on a state f as the composition

$$\mathcal{O} : f \mapsto \mathcal{O}_1 \circ f \circ \mathcal{O}_2^t, \tag{2.29}$$

where t denotes transposition.

Type I Vertex operators

The type I vertex operators, which will now be denoted by $\Phi_{\Lambda_i}^{\Lambda_{1-i}}(z)$, are $U_q(\widehat{sl}_2)$ intertwiners:

$$\Phi_{\Lambda_i}^{\Lambda_{1-i}}(z) : V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_z, \tag{2.30}$$

expressed in terms of components by

$$\Phi_{\Lambda_i}^{\Lambda_{1-i}}(z) = \sum_{\varepsilon=0,1} \Phi_{\Lambda_i,\varepsilon}^{\Lambda_{1-i}} \otimes v_\varepsilon, \tag{2.31}$$

$$\Phi_{\Lambda_i,\varepsilon}^{\Lambda_{1-i}} = \sum_{n\in\mathbb{Z}} \Phi_{\Lambda_i,\varepsilon n}^{\Lambda_{1-i}} z^{-n}, \tag{2.32}$$

where the components are linear maps

$$\Phi_{\Lambda_i,\varepsilon n}^{\Lambda_{1-i}} : V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}).$$

We adopt the normalisation

$$\langle\Lambda_1|\, \Phi_-(z)\, |\Lambda_0\rangle = 1, \quad \langle\Lambda_0|\, \Phi_+(z)\, |\Lambda_1\rangle = 1. \tag{2.33}$$

Type II Vertex operators

We also introduce analogous objects, $\Psi_{\Lambda_i}^{\Lambda_{1-i}}(z)$, called type II vertex operators. These do not have a description in the diagrammatic lattice picture, but will be used in order to create excited states of the spin chain. They are, again, $U_q(\widehat{sl}_2)$ intertwiners

$$\Psi_{\Lambda_i}^{\Lambda_{1-i}}(z) : V(\Lambda_i) \longrightarrow V_z \otimes V(\Lambda_{1-i}), \tag{2.34}$$

with

$$\Psi_{\Lambda_i}^{\Lambda_{1-i}}(z) = \sum_{\varepsilon=0,1} v_\varepsilon \otimes \Psi_{\Lambda_i,\varepsilon}^{\Lambda_{1-i}}, \tag{2.35}$$

$$\Psi_{\Lambda_i,\varepsilon}^{\Lambda_{1-i}} = \sum_{n\in\mathbb{Z}} \Psi_{\Lambda_i,\varepsilon n}^{\Lambda_{1-i}} z^{-n}, \tag{2.36}$$

where

$$\Psi_{\Lambda_i,\varepsilon n}^{\Lambda_{1-i}} : V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}).$$

Again, we have normalisation given by

$$\langle\Lambda_1|\,\Psi_-(z)\,|\Lambda_0\rangle = 1, \quad \langle\Lambda_0|\,\Psi_+(z)\,|\Lambda_1\rangle = 1. \tag{2.37}$$

Dual vertex operators

We introduce the dual vertex operators of type I and type II, which are $U_q(\widehat{sl}_2)$ intertwiners of the form

$$\begin{aligned}\Phi^{*\Lambda_{1-i}}_{\Lambda_i}(z) &: V(\Lambda_i)\otimes V_z \longrightarrow V(\Lambda_{1-i})\\ \Psi^{*\Lambda_{1-i}}_{\Lambda_i}(z) &: V_z\otimes V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}).\end{aligned}$$

As we may expect, normalisation of the components is again chosen in [23] as

$$\begin{aligned}&\langle\Lambda_1|\,\Phi^*_+(z)\,|\Lambda_0\rangle = 1, \quad \langle\Lambda_0|\,\Phi^*_-(z)\,|\Lambda_1\rangle = 1\\ &\langle\Lambda_1|\,\Psi^*_+(z)\,|\Lambda_0\rangle = 1, \quad \langle\Lambda_0|\,\Psi^*_-(z)\,|\Lambda_1\rangle = 1.\end{aligned}$$

We have the following relations for the vertex operators and their duals:

$$\Phi^*_\varepsilon(z) = \Phi_{-\varepsilon}(-q^{-1}z) \tag{2.38}$$

$$\Psi_\varepsilon(z) = \Psi^*_{-\varepsilon}(-q^{-1}z) \tag{2.39}$$

$$g\sum_\varepsilon \Phi^*_\varepsilon(z)\Phi_\varepsilon(z) = 1 \tag{2.40}$$

$$g\Phi_{\varepsilon_1}\Phi^*_{\varepsilon_2}(z) = \delta_{\varepsilon_1\varepsilon_2}, \tag{2.41}$$

where $g = \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}$.

Translation operator

The translation operator translates us by one site along the lattice and is defined in terms of vertex operators by

$$T = g \sum_{\varepsilon} \Phi_{\varepsilon}(1) \otimes \Phi_{-\varepsilon}(1)^t.$$

The inverse of the translation operator is

$$T^{-1} = g \sum_{\varepsilon} \Phi^*_{\varepsilon}(1) \otimes \Phi^*_{-\varepsilon}(1)^t.$$

Local operators

We want to compute correlation functions and form-factors of local operators built from unit matrices $E^{\varepsilon'}_{\varepsilon}$, where this denotes the matrix with 1 in row ε, column ε' and 0 elsewhere. If the components of our tensor product are labelled from the middle as $0, -1, \ldots$ for the right side and as $1, 2, \ldots$ on the left side, i.e. we have

$$\ldots \otimes V_2 \otimes V_1 \otimes V_0 \otimes V_{-1} \otimes V_{-2} \otimes \ldots\ ,$$

then such operators acting on site 1 are defined in terms of vertex operators by

$$E^{\varepsilon'}_{\varepsilon} = g\Phi^*_{\varepsilon}(1)\Phi_{\varepsilon'}(1) \otimes \mathrm{id}. \tag{2.42}$$

The Pauli spin matrices $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ and σ^z are expressed in terms of these unit matrices as

$$\sigma^{\pm}_1 = E^{\mp}_{\pm}, \quad \sigma^z_1 = E^+_+ - E^-_-.$$

Acting on any site n of the chain, we use the translation operator to write

$$\sigma^{\alpha}_n = T^{-(n-1)} \sigma^{\alpha}_1 T^{n-1}, \quad n \in \mathbb{Z}. \tag{2.43}$$

Transfer matrix

The all important transfer matrix of the model can now be introduced as

$$T(z) = g \sum_{\varepsilon} \Phi_{\varepsilon}(z) \otimes (\Phi_{-\varepsilon}(z))^t, \tag{2.44}$$

along with its inverse

$$T(z)^{-1} = g \sum_{\varepsilon} \Phi^*_{\varepsilon}(z) \otimes \left(\Phi^*_{-\varepsilon}(z)\right)^t.$$

Note that $T(1)$ gives the translation operator T, as we would expect. Using the action (2.29) for a linear operator on a state vector $f \in \mathrm{End}(\mathcal{H})$, the transfer matrix acts as

$$T(z)f = g \sum_{\varepsilon} \Phi_{\varepsilon}(z) \circ f \circ \Phi_{-\varepsilon}(z).$$

Vacuum

The *vacuum* in the ith ground state sector is denoted by $|\mathrm{vac}\rangle^{(i)}$ and is the eigenvector of the transfer matrix $T(z)$ with eigenvalue 1. Explicitly,

$$\begin{aligned} T(z)\,|\mathrm{vac}\rangle^{(i)} &= |\mathrm{vac}\rangle^{(1-i)}, \\ |\mathrm{vac}\rangle^{(i)} &= \chi^{-\frac{1}{2}}(-q)^{D^{(i)}} P^{(i)} \in \mathcal{F}, \end{aligned} \tag{2.45}$$

where χ is the character (2.13) at $x = q^2$, i.e.

$$\chi = \chi_{\Lambda_i}(q^{2D^{(i)}}),$$

and $P^{(i)}$ is the projection onto the highest weight module,

$$\mathcal{H} \longrightarrow V(\Lambda_i).$$

If we are considering a left vacuum as an element of the dual space $\mathcal{F}^*$, then we instead write ${}^{(i)}\langle\text{vac}| \in \mathcal{F}^*$.

Excited states

Excited states can be constructed by acting on the vacuum with the type II vertex operators. We introduce spinon spectral parameters $\xi_1, \dots, \xi_n$ with $|\xi_j| = 1$ and define n-particle states by

$$\begin{aligned} |\xi_1, \dots, \xi_n\rangle^{(i)}_{\varepsilon_n,\dots,\varepsilon_1} &= g^{-\frac{n}{2}}\chi^{-\frac{1}{2}}\Psi^*_{\varepsilon_n}(\xi_n)\dots\Psi^*_{\varepsilon_1}(\xi_1)(-q)^{D^{(i)}} && (2.46) \\ {}^{(i)}_{\varepsilon_1,\dots,\varepsilon_n}\langle\xi_1, \dots, \xi_n| &= g^{-\frac{n}{2}}\chi^{-\frac{1}{2}}(-q)^{D^{(i)}}\Psi_{\varepsilon_1}(\xi_1)\dots\Psi_{\varepsilon_n}(\xi_n). && (2.47) \end{aligned}$$

We now introduce the commutation relation of type I and type II vertex operators [21–23]

$$\Phi(z_1)\Psi^*(z_2) = \tau(z_1/z_2)\Psi^*(z_2)\Phi(z_1), \quad \tau(z) = z^{-1}\frac{\Theta_{q^4}(qz^2)}{\Theta_{q^4}(qz^{-2})}, \tag{2.48}$$

where we define the theta function by

$$\Theta_p(x) = (x;p)_\infty(p/x;p)_\infty(p;p)_\infty.$$

Using this, we can verify that the excited states (2.47) and (2.46) are eigenstates of $T(z)^2$, T^2 and H_{XXZ}. The eigenvalues for the one-particle states $|\xi\rangle_{\varepsilon,(i)}$ in particular are given by

$$\begin{aligned} T(z)^2\,|\xi\rangle_{\varepsilon,(i)} &= \tau(\xi)^{-2}\,|\xi\rangle_{\varepsilon,(i)} \\ H\,|\xi\rangle_{\varepsilon,(i)} &= \frac{1-q}{q^2}z\frac{d}{dz}\log\tau(z). \end{aligned}$$

2.5.2 Correlation Functions

In the same spirit as for the one point function in Section 2.4, the correlation function of a product of n-local operators can also be constructed diagrammatically in terms of vertex operators, by considering two vertex operators (one upper and one lower) around each site on which one of the local operators acts. In this way, the n-point correlation function in terms of local operators becomes a $2n$-point function in terms of vertex operators. If we then use the dictionary above for all the components of the trace expression obtained, we can compute correlation functions as traces of $U_q(\widehat{sl}_2)$ intertwiners over irreducible highest weight modules. The result is

$$\begin{aligned} &\frac{{}^{(i)}\langle \text{vac}|\, E^{\varepsilon_n;(n)}_{\varepsilon'_n} \ldots E^{\varepsilon_1;(1)}_{\varepsilon'_1}\, |\text{vac}\rangle^{(i)}}{{}^{(i)}\langle \text{vac}|\text{vac}\rangle^{(i)}} \\ = \; &\frac{\mathrm{Tr}_{V(\Lambda_i)}\left(q^{2D^{(i)}}\Phi_{-\varepsilon'_1}(-q^{-1}z_1)\ldots\Phi_{-\varepsilon'_n}(-q^{-1}z_n)\Phi_{\varepsilon_n}(z_n)\ldots\Phi_{\varepsilon_1}(z_1)\right)}{\mathrm{Tr}_{V(\Lambda_i)}\left(q^{2D^{(i)}}\right)}, \end{aligned} \tag{2.49}$$

where $E^{\varepsilon';(n)}_{\varepsilon}$ denotes local matrix operator $E^{\varepsilon'}_{\varepsilon}$ acting on the nth site of the chain.

2.5.3 Form-Factors

Following the discussion regarding the construction of excited states using type II vertex operators, form-factors of local operators are built in the same way as the corresponding correlation functions, but with the insertion of m type II vertex operators to obtain the m-particle contribution to the form-factor. We have

$$\begin{aligned} &\frac{{}^{(i)}\langle \text{vac}|\, E^{\varepsilon_n;(n)}_{\varepsilon'_n} \ldots E^{\varepsilon_1;(1)}_{\varepsilon'_1}\, |\xi_1,\ldots\xi_n\rangle^{(i)}_{\mu_n,\ldots,\mu_1}}{{}^{(i)}\langle \text{vac}|\text{vac}\rangle^{(i)}} \\ = \; &\frac{\mathrm{Tr}_{V(\Lambda_i)}\left(q^{2D^{(i)}}\Phi_{-\varepsilon'_1}(-q^{-1}z_1)\ldots\Phi_{-\varepsilon'_n}(-q^{-1}z_n)\Phi_{\varepsilon_n}(z_n)\ldots\Phi_{\varepsilon_1}(z_1)\Psi^*_{\mu_n}(\xi_n)\ldots\Psi^*_{\mu_1}(\xi_1)\right)}{\mathrm{Tr}_{V(\Lambda_i)}\left(q^{2D^{(i)}}\right)}. \end{aligned} \tag{2.50}$$

2.6 Bosonization

In order to use these expressions given in terms of purely algebraic objects to make explicit computations, we must introduce a free field realisation of $U_q(\widehat{sl}_2)$. We express our vertex operators in terms of bosonic fields and our highest weight modules in terms of the associated Fock spaces. The trace expressions (2.49) and (2.50) then become traces of bosonic fields over their Fock spaces - something readily calculable using the formula derived in Appendix B. It is within the free field realisation that things become most notably different and technically challenging when we move to the higher spin case. This short section will summarise the Frenkel-Jing free field realisation [77] used by Jimbo and Miwa in [23] for the spin-$\frac{1}{2}$, level one case.

We recall Drinfeld's realisation of $U_q'(\widehat{sl}_2)$ and the relation between the a_n $(n \neq 0)$ generators with $\gamma = q$, as we are working with level one:

$$[a_n, a_m] = \delta_{n+m,0}\frac{[2n][n]}{n}, \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Up to some normalisation, these look like free bosons and they are exactly the operators from which free field realisations of the vertex operators are constructed. We introduce the associated Fock space, $\mathcal{F}^a = \mathbb{C}[a_{-1}, a_{-2}, \ldots]$ and the lattice space $\mathcal{F}^{(i)} = \oplus_{n\in\mathbb{Z}}\mathbb{C}e^{\Lambda_i + n\alpha}$, where $\alpha = \alpha_1$ is the $U_q(\widehat{sl}_2)$ simple root. Zero modes ∂ and α satisfy commutation relation

$$[\partial, \alpha] = 0.$$

For $f \in \mathcal{F}^{(i)}$ and $\beta = \Lambda_i + n\alpha$, we define the action of $a_k, k \neq 0$ on the tensor product space $F^a \otimes \mathcal{F}^{(i)}$ by

$$\begin{aligned} a_k\left(f \otimes e^{\beta}\right) &= a_k f \otimes e^{\beta}, \quad k < 0 \\ &= [a_k, f] \otimes e^{\beta}, \quad k > 0. \end{aligned}$$

The zero mode operators e^{α} and ∂ act on the same space as

$$\begin{aligned} e^{\alpha}\left(f \otimes e^{\beta}\right) &= f \otimes e^{\beta+\alpha} \\ \partial\left(f \otimes e^{\beta}\right) &= (\alpha, \beta) f \otimes e^{\beta}. \end{aligned}$$

The other Drinfeld generators have free field realisation

$$K = q^{\partial}, \quad \gamma = q \tag{2.51}$$
$$X^{\pm} = \exp\left\{\pm\sum_{n=1}^{\infty}\frac{a_{-n}}{[n]}q^{\mp n/2}z^{n}\right\}\exp\left\{\mp\sum_{n=1}^{\infty}\frac{a_{n}}{[n]}q^{\mp n/2}z^{-n}\right\}e^{\pm\alpha}z^{\pm\partial} \tag{2.52}$$
$$q^{d}(1\otimes e^{\beta}) = q^{-(\beta,\beta)/2+i/4}(1\otimes e^{\beta}), \tag{2.53}$$

where $X^{\pm}$ are the Drinfeld currents

$$X^{\pm}(w) = \sum_{n\in\mathbb{Z}} x_n z^{n-1}. \tag{2.54}$$

With this set up, our irreducible highest weight module $V(\Lambda_i)$ has free field realisation

$$V(\lambda_i) = \mathbb{C}[a_{-1}, a_{-2}, \ldots]\otimes\left(\oplus_{n\in\mathbb{Z}}\mathbb{C}e^{\Lambda_i+n\alpha}\right), \tag{2.55}$$

with highest weight vector $|\Lambda_i\rangle = 1\otimes e^{\Lambda_i}$. To determine the explicit form of the components of the vertex operators, we exploit the quantum group symmetry of the model and use the intertwining relation

$$\Delta(x)\circ\Phi(z) = \Phi(z)\circ x, \;\; x\in U_q(\widehat{sl}_2). \tag{2.56}$$

Using (2.12) and (2.56), we obtain the following set of relations between components of the type I vertex operators [23, 82].

$$\begin{aligned}
K\Phi_\pm(z)K^{-1} &= q^{\mp 1}\Phi_\pm(z) \\
q^{-d}\Phi_\pm(z)q^d &= \Phi_\pm(qz) \\
\Phi_+(z)x_0^- - q^{-1}x_0^-\Phi_+(z) &= 0, \\
\Phi_-(z)x_0^- - qx_0^-\Phi_-(z) &= \Phi_+(z), \\
\Phi_+(z)x_0^+ - x_0^+\Phi_+(z) &= K\Phi_-(z), \\
\Phi_-(z)x_0^+ - x_0^+\Phi_-(z) &= 0, \\
\Phi_+(z)x_{-1}^+ - x_{-1}^+\Phi_+(z) &= (qzK)^{-1}\Phi_-(z), \\
\Phi_-(z)x_{-1}^+ - x_{-1}^+\Phi_-(z) &= 0, \\
\Phi_+(z)x_1^- - qx_1^-\Phi_+(z) &= 0, \\
\Phi_-(z)x_1^- - q^{-1}x_1^-\Phi_-(z) &= q^2z\Phi_+(z).
\end{aligned}$$

We are able to choose one vertex operator to have a ‘simple’ definition through these relations. In this case we choose the simple component to be $\Phi_-(z)$, as in [23], which is defined uniquely [85] (up to normalisation) by the above set of intertwining relations to be

$$\Phi_-^{(1-i,i)}(z) = \exp\left\{\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{7n/2}z^n\right\}\exp\left\{-\sum_{n=1}^{\infty}\frac{a_n}{[2n]}q^{-5n/2}z^{-n}\right\}e^{\alpha/2}(-q^3z)^{(\partial+i)/2}.$$

For a detailed derivation of this, see Section 6.2 of [23]. The other component is then given in terms of $\Phi_-(z)$ through

$$\Phi_-(z)x_0^- - qx_0^-\Phi_-(z) = \Phi_+(z).$$

We consider the series expansion for the Drinfeld currents (2.54) and use contour integration to pull out the -1-th w coefficient, writing

$$x_0^\pm = \oint \frac{dw}{2\pi i}\frac{1}{w}X^\pm(w),$$

so that in terms of X^-, we have

$$\Phi_+(z) = \oint \frac{dw}{2\pi i} \left\{ \Phi_-(z) X^-(w) - q X^-(w) \Phi_-(z) \right\}.$$

This process is the origin of the contour integrals that appear in the final expressions for correlation functions and form-factors using the vertex operator approach [22, 23, 82].

2.6.1 Normal Ordering Notation

In order to use the bosonic trace formula (3.35), we first have to *normal order* the bosonic fields and zero modes over which we are taking the trace. This amounts to taking all of the creation operators to the left of the product and all of the annihilation operators to the right. We denote the normal ordered product of bosonic operators $\mathcal{O}_1$ and $\mathcal{O}_2$ by $:\mathcal{O}_1\mathcal{O}_2:$ and their normal ordering factor by $N_{\mathcal{O}_1\mathcal{O}_2}$, where

$$\mathcal{O}_1\mathcal{O}_2 \quad = \quad N_{\mathcal{O}_1\mathcal{O}_2} :\mathcal{O}_1\mathcal{O}_2: \ .$$

In terms of bosons a_n and zero modes δ, α, the normal ordered products are

$$\begin{aligned} :a_n a_{-n}: \quad &= \quad :a_{-n} a_n: = a_{-n} a_n, \qquad n > 0, \\ :\delta\alpha: \quad &= \quad :\alpha\delta: = \alpha\delta. \end{aligned}$$

The vacuum-vacuum matrix element of a pure normal ordered product of operators therefore vanishes, whilst the same matrix element of an exponential of a pure normal ordered product of operators is the identity, greatly simplifying computations.

If we have more than two exponentiated bosonic operators to normal order, we are able to express their combined normal ordering factor as a product of pairwise normal ordering factors[5]. For example, given n bosonic operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$, we

[5] This is because of a drastic simplification in Wick's theorem [86] for this situation.

have

$$e^{\mathcal{O}_1} e^{\mathcal{O}_2} \dots e^{\mathcal{O}_n} \quad = \quad \prod_{i<j} \mathcal{N}_{\mathcal{O}_i \mathcal{O}_j} \, : e^{\mathcal{O}_1} e^{\mathcal{O}_2} \dots e^{\mathcal{O}_n} : ,$$

where $\mathcal{N}_{\mathcal{O}_i \mathcal{O}_j} = \exp \left\{ N_{\mathcal{O}_i \mathcal{O}_j} \right\}$. The ability to express products of operators in this way is extremely useful in the subsequent chapters.

We now move to Chapter 2, where the spin-1 analogue of the techniques discussed in this background section are applied to the spin-1 XXZ chain.

Chapter 3

The One Boson, One Fermion Approach

Having discussed the formalism of the vertex operator approach, we will now consider the higher spin analogue of this with respect to the antiferromagnetic spin-1 XXZ chain. Just as the 6-vertex model is inherently linked with the spin-$\frac{1}{2}$ chain, the spin-1 chain has the 19-vertex model as its associated lattice model [87], and the quantum Hamiltonian is given by the following [50, 87, 88].

$$
\begin{aligned}
\mathcal{H}_{XXZ} = J \sum_{i=-\infty}^{\infty} \Bigg[& S_i \cdot S_{i+1} - (S_i \cdot S_{i+1})^2 \\
& + \frac{1}{2}(q - q^{-1})^2 \left\{ S_i^z \cdot S_{i+1}^z - (S_i^z \cdot S_{i+1}^z)^2 + 2(S_i^z)^2 \right\} \\
& - (q + q^{-1} - 2) \left(S_i^x \cdot S_{i+1}^x + S_i^y \cdot S_{i+1}^y \right) \cdot S_i^z \cdot S_{i+1}^z \\
& + (q + q^{-1} - 2) S_i^z \cdot S_{i+1}^z \cdot \left(S_i^x \cdot S_{i+1}^x + S_i^y \cdot S_{i+1}^y \right) \Bigg],
\end{aligned}
\tag{3.1}
$$

where S^x, S^y and S^z are the standard 3×3 spin matrices. We also introduce $S^{\pm} = S^x \pm iS^y$ as we will later consider S^+ form-factors in particular. As in the case of the spin-$\frac{1}{2}$ chain, we consider the massive region $|q| < 1$. The infinite chain (3.1) has quantum affine symmetry $U_q(\widehat{sl}_2)$ [21, 89] and so we are able to apply the vertex operator approach. We do this with a view to computing form-factors of local operators $E_{\varepsilon_2}^{\varepsilon_1}$ with $\varepsilon \in \{0, 1, 2\}$, where $E_{\varepsilon_1}^{\varepsilon_2}$ denotes the 3×3

matrix with entry 1 in row ε_1, column ε_2 and 0s elsewhere. We use a level-two free field realisation of $U_q(\widehat{sl}_2)$ and build local operators from spin-1, type I vertex operators. Our excited states are still constructed by acting on highest weight states with spin-$\frac{1}{2}$ type II vertex operators. For example, we may like to consider the two-particle form-factor

$$ {}^{(i)}\langle \mathrm{vac}| E^{\varepsilon_2}_{\varepsilon_1} |\xi_1, \xi_2\rangle^{(i;\pm)}_{\ell_1,\ell_2} \tag{3.2} $$

in which the choice of $i \in \{0, 1, 2\}$ corresponds to the 3 possible ground states

$$ \cdots 020202 \cdots , \quad \cdots 111111 \cdots , \quad \cdots 202020 \cdots $$

respectively, ξ is a continuous spinon spectral parameter, $\ell \in \{0, 1\}$ denotes the spin of a spinon, and $\pm$ is an extra degree of freedom indicating the kink nature $(i, i \pm 1, i)$ of the pair of excitations, i.e. whether they step us up or down through levels of excitations.

3.1 Perfect/Imperfect Vertex Operators and the Nature of Excitations

We call a vertex operator *perfect* when the spin attached to it is maximal. In practice, this means that a level-k, spin-$\frac{l}{2}$ vertex operator is perfect when $l = k$. For perfect vertex operators, the choice of irreducible highest weight $U_q(\widehat{sl}_2)$-modules that it intertwines is unique and the vertex operators always act from $V(\lambda_m)$ to $V(\lambda_{k-m})$. In the spin-$\frac{1}{2}$ case considered in Chapter 2, we only encountered perfect type I and type II vertex operators. For higher spin models, we may use perfect type I vertex operators in order to construct local operators, but we will always have spin-$\frac{1}{2}$ physical excitations and so our type II vertex operators will be imperfect level-k, spin-$\frac{1}{2}$ intertwiners. Imperfect vertex operators are generally of a more complicated nature than their perfect counterparts, as the name may suggest, with more involved commutation relations and free field realisations.

Whilst the excitations are described in terms of $2n$-particle spinons, the $2n$-particle space has an RSOS type structure, as discussed in [21], which is apparent in the commutation relations of the type II vertex operators given in this chapter (3.29). The physical S-matrix of the excitations is described by the 'W' weights of these commutation relations, consistent with the picture proposed by Reshetikhin in [90] and Suzuki in [91]. In [21], the double feature of the excitations (i.e. being built of spinons with some additional RSOS properties) is made explicit by the decomposition of crystals, where the nature of the excitations is split into two types - the 'type' of the domain (i.e. the spin of the spinon) and the 'type' of the domain wall. The nature of the domain wall is what we would describe as the 'kink' nature of 2-particle spinon excitation.

3.2 Trace Expressions for Spin-1 XXZ

Our two types of level-two vertex operator are introduced as the level-two $U_q(\widehat{sl}_2)$ intertwiners

$$\tilde{\Phi}_{\lambda_i}^{\lambda_{2-i}}(z) \quad : V(\lambda_i) \to V(\lambda_{2-i}) \otimes V_z^{(2)}, \qquad \text{Type I}, \tag{3.3}$$

$$\tilde{\Psi}_{\lambda_i}^{\lambda_{i\pm1}}(z) \quad : V(\lambda_i) \to V_z^{(1)} \otimes V(\lambda_{i\pm1}), \qquad \text{Type II}, \tag{3.4}$$

where $\lambda_i = i\Lambda_1 + (2-i)\Lambda_0$, $i \in \{0,1,2\}$ and $V(\lambda_i)$ are the three level-two irreducible highest weight modules. In terms of components, we have

$$\begin{aligned}
\tilde{\Phi}_{\lambda_i}^{\lambda_{2-i}}(z) &= \sum_{m=0}^{2} \tilde{\Phi}_{\lambda_i,m}^{\lambda_{2-i}}(z) \otimes v_m^{(2)} \\
\tilde{\Psi}_{\lambda_i}^{\lambda_{i\pm1}}(z) &= \sum_{m=0,1} v_m^{(1)} \otimes \tilde{\Psi}_{\lambda_i,m}^{\lambda_{i\pm1}}(z),
\end{aligned}$$

where $\left\{v_m^{(l)}|m=0,\ldots,l\right\}$ are the basis vectors of the $l+1$-dimensional vector space $V^{(l)}$. Dual vertex operators are defined by

$$\tilde{\Phi}^{*\lambda_{2-i}}_{\lambda_i}(z) \quad : V(\lambda_{2-i})\otimes V_z^{(2)} \to V(\lambda_i), \qquad \text{Type I,} \tag{3.5}$$

$$\tilde{\Psi}^{*\lambda_{i\pm1}}_{\lambda_i}(z) \quad : V_z^{(1)}\otimes V(\lambda_{i\pm1}) \to V(\lambda_i), \qquad \text{Type II.} \tag{3.6}$$

The commutation relations between the different vertex operators analogous to (2.48) are discussed in detail in Section 3.5 of [21]. We note here that the type II relations, (3.39) of [21], are far more complicated than those for type II. We consider this as a manifestation of their imperfect nature and of the nature of the excitations, as will be further discussed as we defined the vertex operators explicitly. In terms of these operators, we have the following trace expression for the n-point function of an arbitrary local operator, up to normalisation dependent on the spaces between which each vertex operator acts.

$$\begin{aligned}&{}^{(i)}\langle \mathrm{vac}|E^{\varepsilon_1'}_{\varepsilon_1}\ldots E^{\varepsilon_n'}_{\varepsilon_n}|\mathrm{vac}\rangle^{(i)} \propto \\ &\mathrm{Tr}_{V(\lambda_i)}\left(q^{-2\rho}\Phi_{2-\varepsilon_1}(q^{-2}z_1)\ldots\Phi_{2-\varepsilon_n}(q^{-2}z_n)\Phi_{\varepsilon_n'}(z_n)\ldots\Phi_{\varepsilon_1'}(z_1)\right).\end{aligned} \tag{3.7}$$

If we also use the type II vertex operator, we have the following trace expression for the m-particle form factor of an arbitrary local operator.

$$\begin{aligned}&{}^{(i)}\langle \mathrm{vac}|E^{\varepsilon_1'}_{\varepsilon_1}\ldots E^{\varepsilon_n'}_{\varepsilon_n}|\xi_1,\ldots\xi_m\rangle^{(i;\pm)}_{\ell_1,\ldots,\ell_m} \propto \\ &\mathrm{Tr}_{V(\lambda_i)}\left(q^{-2\rho}\Phi_{2-\varepsilon_1}(q^{-2}z_1)\ldots\Phi_{2-\varepsilon_n}(q^{-2}z_n)\Phi_{\varepsilon_n'}(z_n)\ldots\Phi_{\varepsilon_1'}(z_1)\Psi^*_{\ell_m}(\xi_m)\ldots\Psi^*_{\ell_1}(\xi_1)\right).\end{aligned} \tag{3.8}$$

3.3 Free Field Realisation

We use the simplest and most standard free field realisation, discussed in [50] and based on ideas from [92], which uses one boson and one fermion. We follow the conventions of [59], [60] and [61]. We are dealing with level-two modules and so have the following commutation relation between Drinfeld generators a_k:

$$[a_k, a_l] = \delta_{k+l,0}\frac{[2k]^2}{k}.$$

From this, we see that the a_k constitute free bosons (up to normalisation) and it is from these that we will construct the bosonic part of the Fock spaces with which we will identify our highest weight modules.

3.3.1 Fock Spaces

We will construct our desired Fock spaces as the tensor product of a bosonic, a fermionic and a lattice space. To begin with, we define the bosonic Fock space $\mathcal{F}^a$ by

$$\begin{aligned}\mathcal{F}^a &= \mathbb{C}[a_{-1}, a_{-2}, \ldots]\,|0\rangle \\ &= \bigoplus_{\substack{1\le i_1<\ldots<i_s \\ n_1,\ldots,n_s>0}} \mathbb{C}\, a_{-i_1}^{n_1}\ldots a_{-i_s}^{n_{i_s}}\,|0\rangle\,,\end{aligned}$$

where $|0\rangle$ denotes a vacuum vector such that $a_m\,|0\rangle = 0$, $m > 0$. As such, for $m > 0$, our a_{-m} and a_m are creation and annihilation operators, respectively.

To construct level-two irreducible highest weight modules, we need to introduce fermions to the free field realisation. For $V(\lambda_0) = V(2\Lambda_0)$ and $V(\lambda_2) = V(2\Lambda_1)$, we use a Neveu-Schwarz fermion and for $V(\lambda_1) = V(\Lambda_0 + \Lambda_1)$, we use a Ramond fermion. This follows the same idea as in the construction of level-two modules over $\widehat{sl}_2$ [59, 60, 92]. We introduce the Neveu-Schwarz fermion ϕ_n^{NS}:

$$\{\phi_n^{NS}|n \in \mathbb{Z} + 1/2\},$$

and the Ramond fermion:

$$\{\phi_n^{R}|n \in \mathbb{Z}\}.$$

We have fermion fields in each case, defined by

$$\phi^{NS}(z) = \sum_{n\in\mathbb{Z}+1/2} \phi^{NS}_n z^{-n}, \tag{3.9}$$

$$\phi^{R}(z) = \sum_{n\in\mathbb{Z}} \phi^{R}_n z^{-n}. \tag{3.10}$$

Both types of fermion satisfy the anti-commutation relation

$$\{\phi^X_m, \phi^X_n\} = \delta_{m+n,0}\frac{q^{2m}-q^{-2m}}{q+q^{-1}}, \quad X = NS, R.$$

For $n > 0$, ϕ^X_n and ϕ^X_{-n} are annihilation and creation operators, respectively. We denote the two corresponding vacua by $|NS\rangle$ and $|R\rangle$, so that

$$\begin{aligned} \phi^{NS}_n |NS\rangle &= 0, \quad n > 0, \\ \phi^{R}_n |R\rangle &= 0, \quad n > 0. \end{aligned}$$

We note that ϕ^R_0 is special as it anti-commutes with the other Ramond fermions and satisfies $\phi^R_0 |R\rangle = |R\rangle$. Our fermion Fock spaces are constructed from these fermions as

$$\begin{aligned} \mathcal{F}^{\phi^{NS}} &= \mathbb{C}[\phi_{-1/2}, \phi_{-3/2}, \ldots] |NS\rangle, \\ \mathcal{F}^{\phi^{R}} &= \mathbb{C}[\phi_{-1}, \phi_{-2}, \ldots] |R\rangle. \end{aligned}$$

We can split our fermion Fock spaces into sub sectors $\mathcal{F}^{\phi^X}_{\text{even/odd}}$ consisting of only even or odd particle states as indicated.

Finally, we need to define the necessary lattice spaces. We let $Q = \mathbb{Z}\alpha$ be the root lattice of Lie algebra sl_2 with group algebra $\mathbb{C}[Q]$. The vector space $\mathbb{C}[Q]$ is spanned by elements $e^{n\alpha}$, $n \in \mathbb{Z}$. Acting on this space, we have bosonic zero modes e^β, $\beta \in \mathbb{Z}\alpha$ and ∂, where

$$[\partial, \alpha] = 2,$$

and

$$\begin{aligned} e^{\beta_1} \cdot e^{\beta_2} &= e^{\beta_1+\beta_2}, \quad \beta_1 \in \mathbb{Z}\alpha, \beta_2 \in \mathbb{Z}\frac{\alpha}{2}, \\ e^{\partial} e^{n\alpha} &= e^{n[\partial,\alpha]} e^{n\alpha} e^{\partial}. \end{aligned}$$

We now have all of the ingredients needed to introduce what we will call the total Fock spaces, $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$, defined by

$$\mathcal{F}^{(0)} \equiv \mathcal{F}^a \otimes \mathcal{F}^{\phi^{NS}} \otimes \mathbb{C}[Q], \tag{3.11}$$

$$\mathcal{F}^{(1)} \equiv \mathcal{F}^a \otimes \mathcal{F}^{\phi^{R}} \otimes e^{\frac{\alpha}{2}} \mathbb{C}[Q]. \tag{3.12}$$

The bosons, fermions and bosonic zero modes act naturally on their associated Fock space and trivially on others.

3.3.2 Drinfeld Generators and Highest Weight Modules

The action of the Drinfeld generators γ and K in the free field realisation is defined as

$$\gamma = q^2, \quad K = q^{\partial}. \tag{3.13}$$

We will also need the following operators, given in terms of Drinfeld generators a_n:

$$\begin{aligned} E^{\pm}_{<}(z) &= \exp\left(\pm \sum_{n=1}^{\infty} \frac{a_{-n}}{[2n]} q^{\mp n} z^n \right), \\ E^{\pm}_{>}(z) &= \exp\left(\mp \sum_{n=1}^{\infty} \frac{a_{n}}{[2n]} q^{\mp n} z^{-n}. \right). \end{aligned}$$

We are now able to give the free field realisation of the Drinfeld currents $x^{\pm}$ and our $U'_q(\widehat{sl}_2)$-modules. The explicit form of the currents is chosen so that, as operators on the total Fock space $\mathcal{F}^{(i)}$, they generate irreducible module $V(\lambda_i)$ by acting on highest weight vector $|\lambda_i\rangle$. In terms of $E^{\pm}_{<}$ and $E^{\pm}_{>}$, we have

$$x^{\pm}(w) = E^{\pm}_{<}(w)\, E^{\pm}_{>}(w)\, \phi(w) e^{\pm\alpha} w^{(1\pm\partial)/2}, \tag{3.14}$$

where $\phi(w)$ is either the Neveu-Schwarz fermion field (3.9) or Ramond fermion field (3.10), depending on the total Fock space we are acting on. With these definitions, we have given the action of the algebra $U'_q(\widehat{sl}_2)$ on $\mathcal{F}^{(i)}$, $i = 0, 1$ and so our total Fock spaces are $U'_q(\widehat{sl}_2)$-modules.

Irreducible highest weight module $V(2\Lambda_0)$

Consider the vector $|\lambda_0\rangle = |0\rangle \otimes |NS\rangle \otimes 1 \in \mathcal{F}^{(0)}$. This generates the $U'_q(\widehat{sl}_2)$-module

$$\mathcal{F}^{(0)}_{+} = \mathcal{F}^{a} \otimes \left\{ \left(\mathcal{F}^{\phi^{NS}}_{\text{even}} \otimes \mathbb{C}[2Q] \right) \oplus \left(\mathcal{F}^{\phi^{NS}}_{\text{odd}} \otimes e^{\alpha} \mathbb{C}[2Q] \right) \right\}.$$

We also have

$$\gamma \cdot |\lambda_0\rangle = q^2\, |\lambda_0\rangle\,, \quad K \cdot |\lambda_0\rangle = |\lambda_0\rangle\,.$$

Recalling our definition of highest weight modules from Chapter 2, we have a level-two module generated by $|\lambda_0\rangle$ which has weight $2\Lambda_0$. Therefore, we have highest weight vector $|\lambda_0\rangle = |2\Lambda_0\rangle$ and identify

$$V(2\Lambda_0) \simeq \mathcal{F}^{a} \otimes \left\{ \left(\mathcal{F}^{\phi^{NS}}_{\text{even}} \otimes \mathbb{C}[2Q] \right) \oplus \left(\mathcal{F}^{\phi^{NS}}_{\text{odd}} \otimes e^{\alpha} \mathbb{C}[2Q] \right) \right\}. \tag{3.15}$$

Irreducible highest weight module $V(2\Lambda_1)$

If we instead look at the vector $|\lambda_2\rangle = |0\rangle \otimes |NS\rangle \otimes e^{\alpha}$, we see that

$$\gamma \cdot |\lambda_2\rangle = q^2\, |\lambda_2\rangle\,, \quad K \cdot |\lambda_2\rangle = q^2\, |\lambda_2\rangle\,,$$

and so it has weight $2\Lambda_1$. This vector generates the $U_q'(\widehat{sl}_2)$-module

$$\mathcal{F}_-^{(0)} = \mathcal{F}^a \otimes \left\{ \left(\mathcal{F}_{\text{even}}^{\phi^{NS}} \otimes e^{\alpha}\mathbb{C}[2Q] \right) \oplus \left(\mathcal{F}_{\text{odd}}^{\phi^{NS}} \otimes \mathbb{C}[2Q] \right) \right\}.$$

We therefore identify

$$V(2\Lambda_1) \simeq \mathcal{F}^a \otimes \left\{ \left(\mathcal{F}_{\text{even}}^{\phi^{NS}} \otimes e^{\alpha}\mathbb{C}[2Q] \right) \oplus \left(\mathcal{F}_{\text{odd}}^{\phi^{NS}} \otimes \mathbb{C}[2Q] \right) \right\}, \tag{3.16}$$

and have highest weight vector $|\lambda_2\rangle = |2\Lambda_1\rangle$.

Irreducible highest weight module $V(\Lambda_0 + \Lambda_1)$

Finally, we need to realise $V(\Lambda_0+\Lambda_1)$ in terms of our free field realisation. In this case, we are working in the Ramond sector and identify the highest weight module with the total Ramond Fock space (3.12),

$$V(\Lambda_0) + V(\Lambda_1) \;\; = \;\; \mathcal{F}^a \otimes \mathcal{F}^{\phi^R} \otimes e^{\frac{\alpha}{2}}\mathbb{C}[Q], \tag{3.17}$$

with highest weight vector $|\lambda_1\rangle = |V(\Lambda_0) + V(\Lambda_1)\rangle = 1 \otimes |R\rangle \otimes e^{\frac{\alpha}{2}}$, as in [61].

It is important to note that the Fock spaces appearing in (3.15), (3.16) and (3.17) are irreducible and so we can identify them with the irreducible highest weight modules in a natural way. As discussed in [50], this makes the task of computing traces over irreducible highest weight modules relatively straightforward as we are able to take the bosonic trace over these Fock spaces. In Chapter 5 we consider the q-Wakimoto bosonisation, where the bosonic Fock space is reducible, and must therefore take its non-trivial BRST cohomology structure into account, as described in the classical case in [93] and Appendix 9.B of [86].

The grading operator

We now define the operator d by

$$d = -\sum_{m=1}^{\infty} mN_m^a - \sum_{k>0} kN_k^{\phi^X} - \frac{1}{8}\partial^2 + \frac{(\lambda, \lambda)}{4}, \tag{3.18}$$

where N_m^a and $N_m^{\phi^X}$ count bosons or fermions by

$$N_m^a = \frac{m}{[2m]^2} a_{-m} a_m, \quad N_m^{\phi^X} = \frac{q + q^{-1}}{q^{2m} + q^{-2m}} \phi_{-m}^X \phi_m^X, \quad m > 0,$$

where $X = NS, R$ and $\lambda = (2 - i)\Lambda_0 + i\Lambda_1$, $i = 0, 1, 2$, depending on which sector we are in. Throughout the following chapters, we sometimes restrict the grading operator to its bosonic and fermionic parts, defining

$$d^a = -\sum_{m=1}^{\infty} mN_m^a, \tag{3.19}$$

$$d^{\phi^X} = -\sum_{k>0} kN_k^{\phi^X}. \tag{3.20}$$

We identify

$$\rho = 2d + \frac{\partial}{2}, \tag{3.21}$$

with the grading operator in $U_q(\widehat{sl}_2)$. With this, along with the identifications (3.15), (3.16) and (3.17), we have the free field realisation of irreducible highest weight modules $V(\lambda_i)$, $i = 0, 1, 2$.

3.3.3 Characters

An explicit formula for characters of level-k irreducible highest-weight modules $V(\lambda_i)$ is given in [94] as

$$\mathrm{Tr}_{V(\lambda_i)}\left(q^{-2\rho}\right) = q^{-i} \frac{\Theta_{q^{2(k+2)}}\left(q^{2(i+1)}\right)}{\Theta_{q^4}\left(q^2\right)}. \tag{3.22}$$

In the present case we have $k = 2$ and so will require characters

$$\mathrm{Tr}_{V(\lambda_i)}\left(q^{-2\rho}\right) = q^{-i}\frac{\Theta_{q^8}\left(q^{2(i+1)}\right)}{\Theta_{q^4}\left(q^2\right)}, \quad i = 0, 1, 2.$$

Using the q-infinite product relations from Appendix A, these simplify to

$$\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right) = (-q^4; q^4)_\infty(-q^2; q^2)_\infty \tag{3.23}$$

$$\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\right) = q^{-1}(-q^2; q^4)_\infty(-q^2; q^2)_\infty \tag{3.24}$$

$$\mathrm{Tr}_{V(\lambda_2)}\left(q^{-2\rho}\right) = q^{-2}(-q^4; q^4)_\infty(-q^2; q^2)_\infty. \tag{3.25}$$

3.3.4 Vertex Operators

It is now time to introduce the free field realisation of the vertex operators (3.3). As in the spin-$\frac{1}{2}$ case considered in [23] and outlined in Section 2.4, it is through the $U_q(\widehat{sl}_2)$ intertwining relations (2.56), that we find explicit free field realisations of these objects. The explicit relations for this case are given in full in [59, 60]. In addition to the fields used to bosonize the Drinfeld currents (3.14), we will require the following operators in order to express our vertex operators explicitly:

$$\begin{aligned}
A_<(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{5n}z^n\right), \\
A_>(z) &= \exp\left(-\sum_{n=1}^{\infty}\frac{a_n}{[2n]}q^{-3n}z^{-n}\right), \\
B_{II,<}(z) &= \exp\left(-\sum_{n=1}^{\infty}\frac{[n]a_{-n}}{[2n]^2}(qz)^n\right), \\
B_{II,>}(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{[n]a_n}{[2n]^2}(q^3z)^{-n}\right).
\end{aligned}$$

Definition 3.1. *Level-two, Spin-1, Type I Vertex Operator*

From [59], we have

$$\begin{aligned}
\Phi_2(z) &= A_<(z)\, A_>(z)\, e^{\alpha}\, (-q^4 z)^{\partial/2}, \\
\Phi_1(z) &= \oint \frac{dw}{2\pi i}\frac{1}{w}(\Phi_2(z)x^-(w) - q^2 x^-(w)\Phi_2(z)) \\
&= -\frac{1-q^4}{q^4 z}\oint \frac{dw}{2\pi i}\frac{1}{w(1-q^{-2}w/z)(1-q^6 z/w)} : \Phi_2(z)x^-(w) :, \\
\Phi_0(z) &= \frac{1}{[2]}\oint \frac{dw}{2\pi i}\frac{1}{w}(\Phi_1(z)x^-(w) - x^-(w)\Phi_1(z)).
\end{aligned}$$

We then normalise according to

$$\tilde{\Phi}_{\lambda_i}^{\lambda_{2-i}}(z)\,|\lambda_i\rangle \;=\; |\lambda_{2-i}\rangle \otimes v_{2-i} + \dots\,, \tag{3.26}$$

where $\dots$ means terms of the form $|\mu\rangle \otimes \nu$, $\mu \neq \lambda_{2-1}$, $\nu \neq v_{2-i}$. From this, we obtain $\tilde{\Phi}_{\lambda_i}^{\lambda_{2-i}}(z) = (-q^4 z)^{i/2}\Phi(z)$. The contours of the integrals are prescribed initially by $|w| = 1$, as we recall from Section 2.6 of the previous chapter that their function is to extract the coefficient of w^{-1} in the series expansion of the currents

$$x^{\pm}(w) = \sum_{n\in\mathbb{Z}} x_n w^{n-1}.$$

We have to also take into account the analyticity regions dictated during the computation of the normal ordering of the currents and vertex operator components. The specific contours for both type I and type II vertex operators will be detailed in Section 3.4 and Section 3.5, respectively.

As discussed, the type II vertex operator required is an imperfect vertex operator, being level-two, but spin-$\frac{1}{2}$. In the current free field realisation, it takes us between Ramond and Neveu-Schwarz sectors. As such, we have to bring in one more type of operator called a fermion emission (or *twist*) operator, $\Omega(z)_X^{X'}$, $(X, X') \in \{(NS, R), (R, NS)\}$ which maps us between the two sectors. These operators appear in [61], [66] and [22] and obey fermion exchange relations[1]

$$\phi^X(w)\Omega(z)^X_{X'} = q^2\left(\frac{z}{w}\right)^{\frac{1}{2}}\frac{(w/q^3z;q^4)_\infty\,(q^7z/w;q^4)_\infty}{(w/qz;q^4)_\infty\,(q^5z/w;q^4)_\infty}\Omega(z)^X_{X'}\phi^X(w). \tag{3.27}$$

The free field realisation of such operators will be considered in more detail later on in the chapter.

Definition 3.2. *Level-two, Spin-$\frac{1}{2}$, Type II Vertex Operator*

From [61], we have

$$\begin{aligned}
\Psi_0(z) &= B_{II,<}(z)\,B_{II,>}(z)\,\Omega(q^{-2}z)\,e^{-\alpha/2}\,(-q^2z)^{-\partial/4},\\
\Psi_1(z) &= \oint\frac{dw}{2\pi i}\frac{1}{w}\left(\Psi_0(z)x^+(w)-qx^+(w)\Psi_0(z)\right)\\
&= \oint\frac{dw}{2\pi i}B_{II,<}(z)\,E^+_<(w)\,B_{II,>}(z)\,E^+_>(w)\Omega(q^{-2}z)\,\phi(w)e^{-\alpha/2}\,(-q^2z/w^2)^{-\partial/4}\\
&\times\ (-q^2zw^3)^{-\frac{1}{2}}\frac{(\frac{w}{qz};q^4)_\infty}{(\frac{qw}{z};q^4)_\infty}\left(\frac{w}{1-q^{-3}w/z}+\frac{q^3z}{1-qz/w}\right).
\end{aligned}$$

We again normalise according to

$$\tilde{\Psi}^{\lambda_{i\pm1}}_{\lambda_i}(z)\,|\lambda_i\rangle \;=\; |\lambda_{i\pm1}\rangle\otimes v_{i\pm1}+\ldots, \tag{3.28}$$

giving $\tilde{\Psi}^{\lambda_j}_{\lambda_i}(z) = g^j_i(z)\,\Psi(z)$, where

$$g^1_0(z)=(-q)^{-1},\quad g^2_1(z)=-(-q^6z)^{-1/4},\quad g^0_1(z)=(-q^2z)^{1/4},\quad g^1_2(z)=(-q^2z)^{1/2}.$$

As discussed, the level II vertex operators obey RSOS type commutation relations, given below.

$$\tilde{\Psi}^{\nu}_{\lambda_{l\pm1},\varepsilon_1}(z_1)\tilde{\Psi}^{\lambda_{l\pm1}}_{\lambda_l,\varepsilon_2}(z_2) \;=\; R^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}(z)\sum_{\mu=\lambda_{l+1},\lambda_{l-1}}\tilde{\Psi}^{\lambda_l}_{\mu,\varepsilon_2'}(z_2)\tilde{\Psi}^{\mu}_{\lambda_l,\varepsilon_1'}(z_1)W\left(\begin{array}{cc|c}\lambda_l & \lambda_{l\pm1} & \\ & & z\\ \mu & \nu & \end{array}\right), \tag{3.29}$$

[1]Note that there is typo in this relation in [61], as discussed in [66].

where $\varepsilon_i = 0, 1$, $l = 0, 1, 2$ and the coefficients $R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}$ are the R-matrix coefficients (recall the R-matrix of the 6-vertex model introduced in Chapter 2):

$$R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(z) = z^{-\frac{1}{2}}\frac{(q^4/z;q^4)_\infty(q^2z;q^4)_\infty}{(q^4z;q^4)_\infty(q^2/z;q^4)_\infty}r^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(z),$$

where

$$\begin{aligned} r^{00}_{00}(z) &= r^{11}_{11}(z) &&= 1 \\ r^{10}_{10}(z) &= r^{01}_{01}(z) &&= q\frac{1-z}{1-q^2z} \\ r^{01}_{10}(z) &= r^{10}_{01}(z) &&= z\frac{1-q^2}{1-q^2z}. \end{aligned}$$

The weight factors W are given by

$$W\left(\begin{array}{cc|c} \lambda_l & \mu & \\ & & z \\ \mu' & \nu & \end{array}\right) = -z^{\Delta_\lambda+\Delta_\nu-\Delta_\mu-\Delta_{\mu'}-\frac{1}{2}}\frac{\xi(z^{-1};1,pq^4)}{\xi(z;1,pq^4)}\hat{W}\left(\begin{array}{cc|c} \lambda_l & \mu & \\ & & z \\ \mu' & \nu & \end{array}\right),$$

where

$$\xi(z,a,b) = \frac{(az;p,q^4)_\infty(a^{-1}bz;p,q^4)_\infty}{(q^2az;p,q^4)_\infty(q^{-2}a^{-1}bz;p,q^4)_\infty},$$

and

$$\triangle_{\lambda_i} = \frac{i(i+2)}{4(k+2)}.$$

The $\hat{W}\left(\begin{array}{cc|c} \lambda_l & \mu & \\ & & z \\ \mu' & \nu & \end{array}\right)$ factors are defined as follows.

$$
\begin{aligned}
\hat{W}\left(\begin{array}{cc|c} \lambda_l & \lambda_{l+1} & \\ & & z \\ \lambda_{l+1} & \lambda_l & \end{array}\right) &= \frac{\Theta_p(pq^2)}{\Theta_p(pq^{-2l-2})}\frac{\Theta_p(pq^{-2l-2}z)}{\Theta_p(pq^2z)}, \\
\hat{W}\left(\begin{array}{cc|c} \lambda_l & \lambda_{l+1} & \\ & & z \\ \lambda_{l-1} & \lambda_l & \end{array}\right) &= q^{-1}\frac{\Gamma_p((2l+1)s)^2}{\Gamma_p((2l+4)s)\Gamma_p(2ls)}\frac{\Theta_p(pz)}{\Theta_p(pq^2z)}, \\
\hat{W}\left(\begin{array}{cc|c} \lambda_l & \lambda_{l-1} & \\ & & z \\ \lambda_{l+1} & \lambda_l & \end{array}\right) &= q^{-1}\frac{\Gamma_p(1-(2l+1)s)^2}{\Gamma_p(1-(2l+4)s)\Gamma_p(1-2ls)}\frac{\Theta_p(pz)}{\Theta_p(pq^2z)}, \\
\hat{W}\left(\begin{array}{cc|c} \lambda_l & \lambda_{l-1} & \\ & & z \\ \lambda_{l-1} & \lambda_l & \end{array}\right) &= z^{-1}\frac{\Theta_p(pq^2)}{\Theta_p(q^{2l+2})}\frac{\Theta_p(q^{2l+2}z)}{\Theta_p(pq^2z)}, \\
\hat{W}\left(\begin{array}{cc|c} \lambda_l & \lambda_{l\pm1} & \\ & & z \\ \lambda_{l\pm1} & \lambda_l^{\pm} & \end{array}\right) &= 1, \\
\hat{W}\left(\begin{array}{cc|c} \lambda_l & \mu & \\ & & z \\ \mu' & \nu & \end{array}\right) &= 0 \quad \text{otherwise,}
\end{aligned}
$$

where $\lambda_l^{\pm} = \lambda_l \pm 2(\Lambda_1 - \Lambda_0)$, $s = \frac{1}{2(k+2)}$, $p = q^{2(k+2)}$ and k is the level of the vertex operator (i.e. $k = 2$ in our case). Finally, $\Gamma_p(z)$ denotes the q-gamma function defined in Appendix A.

These weights can be thought of as the Boltzmann weights in the RSOS face picture and describe the physical S-matrix of the excitations, consistent with the picture of Reshetikhin in [90].

In order to construct form-factors, we will also need the type II dual vertex operators, which are given component-wise in terms of the type II vertex operators by

$$
\begin{aligned}
\tilde{\Psi}^{*\,\lambda_{i+1}}_{\lambda_i,1}(z) &= -q\tilde{\Psi}^{\lambda_{i+1}}_{\lambda_i,0}(q^2z) && (3.30)\\
\tilde{\Psi}^{*\,\lambda_{i+1}}_{\lambda_i,0}(z) &= \tilde{\Psi}^{\lambda_{i+1}}_{\lambda_i,1}(q^2z) && (3.31)\\
\tilde{\Psi}^{*\,\lambda_{i-1}}_{\lambda_i,1}(z) &= \tilde{\Psi}^{\lambda_{i-1}}_{\lambda_i,0}(q^2z) && (3.32)\\
\tilde{\Psi}^{*\,\lambda_{i-1}}_{\lambda_i,0}(z) &= -q^{-1}\tilde{\Psi}^{\lambda_{i-1}}_{\lambda_i,1}(q^2z). && (3.33)
\end{aligned}
$$

3.4 General Formula for the n-point Correlation Function

We now compute a general expression for the trace of an arbitrary product of spin-1, level-two, type I vertex operators

$$\mathrm{Tr}_{V(\lambda_i)}\left(x^{-\rho}\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2)\dots\Phi_{\varepsilon_n}(z_n)\right), \tag{3.34}$$

where $\varepsilon_i \in \{0,1,2\}$ and $\lambda_i = i\Lambda_1 + (2-i)\Lambda_0$. Under specialisations of the variables ε_i and z_i, along with the appropriate normalisation factors, this formula will give us the expression for the n-point correlation function for the spin-1 XXZ chain (3.7).

3.4.1 Boson Contributions

When taking the trace over one of our irreducible highest weight modules, we are able to split the computation into a product of traces over separate bosonic, lattice and fermionic sectors. In our specific case, taking the trace over the bosonic sector $\mathcal{F}^a$ is much simpler and so we will begin here. We first introduce a useful notation of fermion contraction. Given an operator $\mathcal{O}$ expressed in terms of bosonic and fermionic fields, we define the operator $\hat{\mathcal{O}}$ to be the same operator without fermion contributions. For example, looking at

$$x^{\pm}(w) = E^{\pm}_{<}(w)\, E^{\pm}_{>}(w)\, \phi(w) e^{\pm\alpha} w^{(1\pm\partial)/2},$$

we have

$$\hat{x}^{\pm}(w) = E^{\pm}_{<}(w)\, E^{\pm}_{>}(w)\, e^{\pm\alpha} w^{(1\pm\partial)/2}.$$

In order to compute the trace, we first normal order the product of vertex operators using the relations given in Appendix C. On the resultant product of normal ordered operators, we then use the bosonic trace formula,

$$\begin{aligned} &\mathrm{Tr}_{F^a}\left(x^{-\rho}\exp\left(\sum_{n=1}^{\infty}A_n a_{-n}\right)\exp\left(\sum_{n=1}^{\infty}B_n a_n\right)\right) \\ =\ &\exp\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}x^{2mn}A_n B_n[a_n, a_{-n}]\right)\frac{1}{(x^2;x^2)_\infty}, \end{aligned} \tag{3.35}$$

which is derived in Appendix B.

Using fermion contraction along with the normal ordering relations between vertex operator components and currents in Appendix C, we can express the components of the type I vertex operator as

$$\begin{aligned} \Phi_2(z) &= A_<(z)\,A_>(z)\,e^{\alpha}\,(-q^4 z)^{\partial/2}, \\ \Phi_1(z) &= \oint \frac{dw}{2\pi i}\frac{1}{w}(\Phi_2(z)x^-(w) - q^2 x^-(w)\Phi_2(z)), \\ &= \oint \frac{dw}{2\pi i}F_1(z,w) : \Phi_2(z)\hat{x}^-(w) : \phi(w) \\ \Phi_0(z) &= \oint \frac{dv}{2\pi i}\oint \frac{dw}{2\pi i}F_0(z,v,w) : \Phi_2(z)\hat{x}^-(v)\hat{x}^-(w) : \phi(v)\phi(w), \end{aligned}$$

where

$$\begin{aligned} F_1(z,w) &= -\frac{1-q^4}{q^4 z}\frac{1}{w(1-q^{-2}w/z)(1-q^6 z/w)}, \\ & \quad |\frac{w}{q^2 z}| < 1,\ |\frac{q^6 z}{w}| < 1, \end{aligned} \tag{3.36}$$

$$\begin{aligned} F_0(z,v,w) &= \frac{(1-q^2)^2}{q^7 z^2}\frac{1-q^2 w/v}{(1-v/q^2 z)(1-w/q^2 z)(1-q^6 z/v)(1-q^6 z/w)}, \\ & \quad |\frac{v}{q^2 z}| < 1\ |\frac{w}{q^2 z}| < 1,\ |\frac{q^6 z}{v}| < 1\ |\frac{q^6 z}{w}| < 1 \end{aligned} \tag{3.37}$$

and the contours of the integrals are defined by the analyticity regions of (3.11) and (3.12).

We define index sets

$$
\begin{aligned}
\mathcal{A} &= \{a; 1 \leq a \leq n, \varepsilon_a = 0\}, \\
\mathcal{B} &= \{b; 1 \leq b \leq n, \varepsilon_b = 1\}, \\
\mathcal{C} &= \{c, 1 \leq c \leq n, \varepsilon_c = 2\} \\
\mathcal{I} &= \{i; 1 \leq j \leq n\}.
\end{aligned} \tag{3.38}
$$

Using the definitions of the type I vertex operators given above, we are taking the trace of a product built from three possible operators, which we define as

$$
\begin{aligned}
\mathcal{O}_C(z_c) &= \Phi_2(z_c),\ c \in \mathcal{C}, \\
\mathcal{O}_B(z_b, w_b) &= :\Phi_2(z_b)\hat{x}^-(w_b):,\ \ b \in \mathcal{B}, \\
\mathcal{O}_A(z_a, v_a, w_a) &= :\Phi_2(z_a)\hat{x}^-(w_a)\hat{x}^-(v_a):,\ \ a \in \mathcal{A}.
\end{aligned}
$$

These will be ordered in a certain way and appear along with the associated prefactors, integrals and fermionic parts. We denote the bosonic normal ordering factor for each pair of operators by $\mathcal{N}_{X_1X_2}$, so that

$$
\begin{aligned}
\mathcal{O}_{X_1}(z_1, \ldots z_n)\mathcal{O}_{X_2}(w_1, \ldots w_m) &= \mathcal{N}_{X_1X_2}(z_1, \ldots, z_n; w_1, \ldots w_m) \\
&\quad \times :\mathcal{O}_{X_1}(z_1, \ldots z_n)\mathcal{O}_{X_2}(w_1, \ldots w_m): .
\end{aligned}
$$

For every $c \in \mathcal{C}$ and $b \in \mathcal{B}$ such that $c < b$, we have a factor $\Phi_2(z_c)\Phi_1(z_b)$ with operator contribution

$$
\begin{aligned}
\mathcal{O}_C(z_c)\mathcal{O}_B(z_b, w_b) &= \mathcal{N}_{CB}(z_c; z_b, w_b) : \mathcal{O}_C(z_c)\mathcal{O}_B(z_b, w_b) : \\
&= \frac{(1 - q^2 z_b/z_c)}{(1 - w_b/q^2 z_c)} : \mathcal{O}_C(z_c)\mathcal{O}_B(z_b, w_b) : .
\end{aligned} \tag{3.39}
$$

Similarly, for each $c \in \mathcal{C}$ and $b \in \mathcal{B}$ such that $b < c$, we have a contribution

$$
\begin{aligned}
\mathcal{O}_B(z_b, w_b)\mathcal{O}_C(z_c) &= \mathcal{N}_{BC}(z_b, w_b; z_c) : \mathcal{O}_C(z_c)\mathcal{O}_B(z_b, w_b) : \\
&= \frac{-q^4 z_b(1 - q^2 z_c/z_b)}{w_b(1 - q^6 z_c/w_b)} : \mathcal{O}_C(z_c)\mathcal{O}_B(z_b, w_b) : .
\end{aligned} \tag{3.40}
$$

For each $c < a$, we have

$$\begin{aligned}
\mathcal{O}_C(z_c)\mathcal{O}_A(z_a, v_a, w_a) &= \mathcal{N}_{CA}(z_c; z_a, v_a, w_a) : \mathcal{O}_C(z_c)\mathcal{O}_A(z_a, v_a, w_a) : \\
&= \frac{(1 - q^2 z_a/z_c)}{(-q^4 z_c)} \frac{1}{(1 - v_a/q^2 z_c)(1 - w_a/q^2 z_a)} \\
&\quad \times : \mathcal{O}_C(z_c)\mathcal{O}_A(z_a, v_a, w_a) :,
\end{aligned} \tag{3.41}$$

and so on for each possible pair of contributions. The form of each normal ordering factor is detailed in the Appendix C, along with the normal ordering relations for simple vertex operator components and currents. In terms of these normal ordering factors, we then have

$$\begin{aligned}
& \hat{\Phi}_{\varepsilon_1}(z_1)\hat{\Phi}_{\varepsilon_2}(z_2)\dots\hat{\Phi}_{\varepsilon_n}(z_n) \\
= & \prod_a \oint \frac{dv_a}{2\pi i} \oint \frac{dw_a}{2\pi i} F_1(z_a, v_a, w_a) \prod_b \oint \frac{dw_b}{2\pi i} F_0(z_b, w_b) \\
\times & \prod_{a<a'} \mathcal{N}_{AA}(z_a, v_a, w_a; z_{a'}, v_{a'}, w_{a'}) \prod_{b<b'} \mathcal{N}_{BB}(z_b, w_b; z_{b'}, w_{b'}) \\
\times & \prod_{c<c'} \mathcal{N}_{CC}(z_c; z_{c'}) \prod_{c<b} \mathcal{N}_{CB}(z_c; z_b, w_b) \prod_{c>b} \mathcal{N}_{BC}(z_b, w_b; z_c) \\
\times & \prod_{c<a} \mathcal{N}_{CA}(z_c; z_a, v_a, w_a) \prod_{a<c} \mathcal{N}_{AC}(z_a, v_a, w_a; z_c) \\
\times & \prod_{a<b} \mathcal{N}_{AB}(z_a, v_a, w_a; z_b, w_b) \prod_{b<a} \mathcal{N}_{BA}(z_b, w_b; z_a, v_a, w_a) \\
\times & \prod_{\substack{a\in\mathcal{A} \\ b\in\mathcal{B} \\ 1\le i\le n}} : \Phi_2(z_i)\hat{x}^-(w_a)\hat{x}^-(v_a)\hat{x}^-(w_b) :,
\end{aligned}$$

which explicitly looks like

$$
\begin{aligned}
&\hat{\Phi}_{\varepsilon_1}(z_1)\hat{\Phi}_{\varepsilon_2}(z_2)\ldots\hat{\Phi}_{\varepsilon_n}(z_n)\\
= &\prod_a \oint\frac{dv_a}{2\pi i}\oint\frac{dw_a}{2\pi i}F_1(z_a,v_a,w_a)\prod_b\oint\frac{dw_b}{2\pi i}F_0(z_b,w_b)\prod_{a<a'}\frac{(1-q^2z_{a'}/z_a)}{q^{12}z_az_{a'}^2}\\
\times &\frac{w_a^2v_a^2(1-q^2w_{a'}/w_a)(1-q^2v_{a'}/v_a)}{(1-w_{a'}/q^2z_a)(1-v_a/q^2z_a)}\frac{(1-q^2v_{a'}/w_a)(1-q^2w_{a'}/v_a)}{(1-q^6z_{a'}/w_a)(1-q^6z_{a'}/v_a)}\\
\times &\prod_{b<b'}\frac{(1-q^2z_{b'}/z_b)}{q^4z_{b'}}\frac{w_b(1-q^2w_{b'}/w_b)}{(1-w_{b'}/q^2z_b)(1-q^6z_{b'}/w_b)}\prod_{c<c'}\left(1-\frac{q^2z_{c'}}{z_c}\right)(-q^4z_c)\\
\times &\prod_{c<b}\frac{(1-q^2z_b/z_c)}{(1-w_b/q^2z_c)}\prod_{c>b}\frac{z_b(1-q^2z_c/z_b)}{z_c(1-q^6z_c/w_b)}\\
\times &\prod_{c<a}\frac{(1-q^2z_a/z_c)}{q^4z_c}\frac{1}{(1-w_a/q^2z_c)}\frac{1}{(1-v_a/q^2z_c)}\\
\times &\prod_{c>a}\frac{z_a(1-q^2z_c/z_a)}{q^4z_c^2}\frac{1}{(1-q^6z_c/w_a)}\frac{1}{(1-q^6z_c/v_a)}\\
\times &\prod_{a<b}\frac{(1-q^2z_b/z_a)}{q^8z_b^2}\frac{w_av_a(1-q^2w_b/w_a)(1-q^2w_b/v_a)}{(1-w_b/q^2z_a)(1-q^6z_b/v_a)(1-q^6z_b/w_a)}\\
\times &\prod_{a>b}\frac{(1-q^2z_a/z_b)}{z_bz_a}\frac{w_b^2(1-q^2v_a/w_b)(1-q^2w_a/w_b)}{(1-v_a/q^2z_b)(1-w_a/q^2z_b)(1-q^6z_a/w_b)}\\
\times &\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ 1\le i\le m}} :\Phi_2(z_i)\hat{x}^-(w_a)\hat{x}^-(v_a)\hat{x}^-(w_b):\ .
\end{aligned}
$$

It is convenient to abbreviate and define a factor $H(\mathbf{z},\mathbf{v},\mathbf{w})$ by writing this as

$$
\begin{aligned}
&\hat{\Phi}_{\varepsilon_1}(z_1)\hat{\Phi}_{\varepsilon_2}(z_2)\ldots\hat{\Phi}_{\varepsilon_n}(z_n)\\
= &\prod_a \oint\frac{dv_a}{2\pi i}\oint\frac{dw_a}{2\pi i}F_1(z_a,v_a,w_a)\prod_b\oint\frac{dw_b}{2\pi i}F_0(z_b,w_b) \qquad (3.42)\\
&\times H(\mathbf{z},\mathbf{v},\mathbf{w})\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ 1\le i\le m}} :\Phi_2(z_i)\hat{x}^-(w_a)\hat{x}^-(v_a)\hat{x}^-(w_b):\ . \qquad (3.43)
\end{aligned}
$$

We will now focus on computing the trace

$$
\mathrm{Tr}_{V(\lambda_i)}\left(x^{-\rho}\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ 1\le i\le n}} :\Phi_2(z_i)\hat{x}^-(w_a)\hat{x}^-(v_a)\hat{x}^-(w_b):\right). \tag{3.44}
$$

Explicitly, we have

$$\begin{aligned}
\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ 1\leq i\leq m}} :\Phi_2(z_i)\hat{x}^-(w_a)\hat{x}^-(v_a)\hat{x}^-(w_b): &= :\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ i\in\mathcal{I}}} A_<(z_i)E_<^-(w_a)E_<^-(v_a)E_<^-(w_b) \\
&\times \prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ i\in\mathcal{I}}} A_>(z_i)E_>^-(w_a)E_>^-(v_a)E_>^-(w_b): \\
&\times \ e^{-m\alpha}e^{(2t+s)\alpha}\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ i\in\mathcal{I}}}(w_a w_b v_a)^{1/2}\left(\frac{-q^4 z_i}{w_a w_b v_a}\right)^{\partial/2}
\end{aligned}$$

where $s=|\mathcal{B}|$ is the number of Φ_1 components and $t=|\mathcal{A}|$ is the number of Φ_0 components. For our vertex operator $\Phi_2(z)$ and currents $x^-(w)$, we introduce the notation $g_{\mathcal{O}_i\mathcal{O}_j}$ for the trace contributions from a pair of operators $\mathcal{O}_i,\mathcal{O}_j\in\{\Phi_2(z),x^-(w)\}$ and $g_{\mathcal{O}_i}$ for the 'self' contribution from a single operator.

We use the trace formula (3.35) to obtain that for normal ordered operators $:\mathcal{O}_1(z_1)\ldots\mathcal{O}_m(z_m):$ with bosonic part

$$\begin{aligned}
&\exp\left\{\sum_{n>0}\left(A_n^{(1)}(z_1)+\ldots+A_n^{(m)}(z_m)\right)a_{-n}\right\}\\
&\times\exp\left\{\sum_{n>0}\left(B_n^{(1)}(z_1)+\ldots+B_n^{(m)}(z_m)\right)a_{n}\right\},
\end{aligned}$$

we get self contributions

$$g_{\mathcal{O}_i}=\exp\left\{\sum_{n,k=1}^{\infty}A_n^{(i)}(z_i)B_n^{(i)}(z_i)x^{2nk}[a_n,a_{-n}]\right\}, \tag{3.45}$$

and pair-wise contributions

$$\begin{aligned}
g_{\mathcal{O}_i\mathcal{O}_j}(z_i,z_j) &= \exp\left\{\sum_{n,k=1}^{\infty}\left(A_n^{(i)}(z_i)B_n^{(j)}(z_j)+A_n^{(j)}(z_j)B_n^{(i)}(z_i)\right)x^{2nk}[a_n,a_{-n}]\right\}\\
&= g_{\mathcal{O}_j\mathcal{O}_i}(z_j,z_i).
\end{aligned} \tag{3.46}$$

The important trace contributions for our purpose are detailed in Appendix C. We note that, for example, we can write

$$
\begin{aligned}
g_{\Phi_2} g_{\Phi_2} g_{\Phi_2\Phi_2}(z_1, z_2) &= \prod_{i,i'=1}^{2} (x^2 q^2 z_i/z_{i'}; x^2)_\infty \\
&= g_{x^-} g_{x^-} g_{x^- x^-}(z_1, z_2),
\end{aligned}
$$

and define

$$
g_1(z_i, z_j) = (x^2 q^2 z_i/z_j; x^2)_\infty \tag{3.47}
$$

so that we can combine the self contributions and pair-wise contributions of repeated operators with different arguments. With this, the trace expression has the form

$$
\begin{aligned}
&\mathrm{Tr}_{V(\lambda_i)} \left(x^{-\rho} \prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ 1\le i\le m}} \Phi_2(z_i)\hat{x}^-(w_a)\hat{x}^-(v_a)\hat{x}^-(w_b) \right) \\
= \; & \delta_{m,s+2t} \frac{1}{(q^4; q^4)_\infty} \prod_{\substack{a,a'\in\mathcal{A}\\ b,b'\in\mathcal{B}\\ i,i'\in\mathcal{I}}} \oint \frac{dv_a}{2\pi i} \oint \frac{dw_a}{2\pi i} \oint \frac{dw_b}{2\pi i} F_0(z_a, v_a, w_a) F_1(z_b, w_b) \\
& \times g_1(z_i, z_{i'}) g_1(w_a, w_{a'}) g_1(w_b, w_{b'}) g_1(v_a, v_{a'}) \\
& \times g_{x^-x^-}(w_a, v_a) g_{x^-x^-}(w_a, w_b) g_{x^-x^-}(v_a, w_b) \\
& \times g_{\Phi_2 x^-}(z_i, v_a) g_{\Phi_2 x^-}(z_i, w_a) g_{\Phi_2 x^-}(z_i, w_b) \\
& H(\mathbf{z}, \mathbf{v}, \mathbf{w}) \mathrm{Tr}_{V(\lambda_i)} \left(\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ i\in\mathcal{I}}} (w_a w_b v_a)^{1/2} \left(\frac{-q^4 z_i}{w_a w_b v_a} \right)^{\partial/2} \right).
\end{aligned}
$$

The integrals in the expressions obtained in this way always arise through taking the contour integral of the series expansion of Drinfeld currents in the intertwining relations, as discussed in Section 2.6, and so the contours are always prescribed by considering the analyticity regions of the normal ordering factors between each pair of operators. This should be kept in mind throughout, unless other contours are specified.

3.4.2 Lattice Contributions

The lattice space over which we take the trace will differ slightly, depending on our choice of $V(\Lambda_i)$. For $i = 0, 2$, we will get different lattice contributions attached to each different term of the fermionic traces $\mathrm{Tr}_{\mathcal{F}^{\phi^{NS}}_{\text{even}}}$ and $\mathrm{Tr}_{\mathcal{F}^{\phi^{NS}}_{\text{odd}}}$. We recall the free field realisations of the irreducible highest weight modules (3.15), (3.16) and (3.17) and see that we need to take the trace over the lattice spaces $\mathbb{C}[2Q]$, $e^{\alpha}\mathbb{C}[2Q]$ and $e^{\frac{\alpha}{2}}\mathbb{C}[Q]$. For each, we use the basic formula for the trace over the total lattice space [59, 60]

$$\mathrm{Tr}_{\mathbb{C}[Q]}\left(x^{\partial^2/4-\partial/2}f^{\partial}\right) = \sum_{l\in\mathbb{Z}} x^{l^2}x^{-l}f^{2l}. \tag{3.48}$$

In the above, we take $l \mapsto 2l$

$$\mathrm{Tr}_{\mathbb{C}[2Q]}\left(x^{\partial^2/4-\partial/2}f^{\partial}\right) = \sum_{l\in\mathbb{Z}} x^{4l^2-2l}f^{4l}.$$

Next, taking $l \mapsto 2l+1$ in (3.48), we have

$$\mathrm{Tr}_{e^{\alpha}\mathbb{C}[2Q]}\left(x^{\partial^2/4-\partial/2}f^{\partial}\right) = \sum_{l\in\mathbb{Z}} x^{4l^2+2l}f^{4l+2}.$$

Finally, we shift $l \mapsto l+\frac{1}{2}$ in (3.48) to obtain

$$\mathrm{Tr}_{e^{\alpha/2}\mathbb{C}[Q]}\left(x^{\partial^2/4-\partial/2}f^{\partial}\right) = \sum_{l\in\mathbb{Z}} x^{l^2-1/4}f^{2l+1}.$$

Using the identity

$$\Theta_{y^2}(yz) = \sum_{n\in\mathbb{Z}}(-1)^n y^{n^2}z^n,$$

for $x = q^2$ we have the following

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{C}[2Q]}\left(q^{-2\rho}f^{\partial}\right) &= \Theta_{q^{16}}(-q^4f^4), \\
\mathrm{Tr}_{e^{\alpha}\mathbb{C}[2Q]}\left(q^{-2\rho}f^{\partial}\right) &= f^2\Theta_{q^{16}}(-q^{12}f^4), \\
\mathrm{Tr}_{e^{\frac{\alpha}{2}}\mathbb{C}[Q]}\left(q^{-2\rho}f^{\partial}\right) &= fq^{-\frac{1}{2}}\Theta_{q^4}\left(-q^2f^2\right).
\end{aligned}$$

In the case of the n-point function (3.7), our f will be given by $\prod\limits_{a,b,i}\left(\frac{-q^4z_i}{w_aw_bv_a}\right)^{1/2}$, for $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $i \in \mathcal{I}$.

3.4.3 Fermion Contributions

In this section, we consider the fermionic contribution to the n-point function, which will involve taking the trace over an ordered product of fermion fields. For each $\Phi_0(z_a)$, $a \in \mathcal{A}$ we will have a contribution $\phi(v_a)\phi(w_a)$ and for each $\Phi_1(z_b)$, $b \in \mathcal{B}$, we will have a contribution $\phi(w_b)$. Using results from [60] for the trace of a product of fermions, we do not need to normal order the fields, but we do need to keep track of their order and so introduce the function

$$h(v_k, w_k) = \begin{cases} \phi(v_k)\phi(w_k) & k \in \mathcal{A} \\ \phi(w_k), & k \in \mathcal{B} \\ 1, & k \in \mathcal{C}, \end{cases},$$

which allows the ordered fermion product to be written as

$$\overset{\hookrightarrow}{\prod_{k=1}^{n}} h(v_k, w_k),$$

where we introduce ordered product notation

$$\overset{\hookrightarrow}{\prod_{k=1}^{n}} f(k) = f(1)f(2)\dots f(n).$$

In order to use Idzumi's expression for the trace of such an ordered product of fermions, it is convenient to relabel the arguments of the fields and write instead

$$\overset{\hookrightarrow}{\prod_{k=1}^{n}} h(v_k, w_k) \;=\; \overset{\hookrightarrow}{\prod_{j=1}^{s+2t}} h(v_k, w_k)\phi(u_j), \tag{3.49}$$

where again, $s = |\mathcal{B}|$ and $t = |\mathcal{A}|$. In computing a specific trace, it will be important to keep track of the relabelling so that we can return to the original arguments before inserting this into the integral expression for the total trace. Before stating Idzumi's result, we define the Pfaffian of a matrix.

Definition 3.3. *Pfaffians*

The determinant of any skew symmetric matrix A can always be written as the square of a polynomial in its entries. This polynomial is called the Pfaffian of A and is only non zero for $2n$ by $2n$ matrices. Formally, given a $2n$ by $2n$ matrix A with entries a_{ij}, the Pfaffian of A is given by the expression

$$\mathrm{Pf}(A) \;=\; \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma_{2i-1},\sigma_{2i}}, \tag{3.50}$$

with S_{2n} defined as the symmetric group on $2n$ letters generated by $\{\sigma_1, \ldots, \sigma_{2i-1}\}$, where the generator σ_i permutes letters i and $i+1$. The signature of a permutation σ, denoted $\mathrm{sgn}(\sigma)$, is defined to be 1 or -1 if σ is an even or odd permutation, respectively.

Fermionic Traces as Pfaffians

As discussed in [60], the trace

$$T_\phi(w_1, \ldots, w_n) = \mathrm{Tr}_{\mathcal{F}^\phi}\left(x^{-2d^\phi}\phi(w_1)\ldots\phi(w_n)\right)$$

can be expressed as the Pfaffian of a matrix, G, with entries built from two point functions of the fermion fields:

$$T_\phi(w_1,\dots,w_n) = \begin{cases} \mathrm{Tr}\left(x^{-2d^\phi}\right) \times \mathrm{Pf}(G(w_1,\dots,w_n)), & n \quad \text{even} \\ \mathrm{Tr}\left(x^{-2d^\phi}\right) \times \mathrm{Pf}(\bar{G}(w_i,\dots,w_n)), & n \quad \text{odd}, \phi = \phi^R \\ 0, & \text{otherwise} \end{cases} . \quad (3.51)$$

For n even, and $\phi = \phi^{NS}$, $G(w_1,\dots,w_n)$ is an $n\times n$ antisymmetric matrix with entries given by two point functions

$$\begin{aligned} G^{NS}_{ij}(w_1,\dots,w_n) &= G^{NS}(w_i,w_j) \\ &= \frac{\mathrm{Tr}\left(x^{-2d^{\phi^{NS}}}\phi^{NS}(w_i)\phi^{NS}(w_j)\right)}{\mathrm{Tr}\left(x^{-2d^{\phi^{NS}}}\right)} \\ &= \frac{1}{q+q^{-1}}\sum_{m\in\mathbb{Z}}\left(\frac{w_j}{w_i}\right)^m \frac{q^{2m}+q^{-2m}}{1+x^{2m}}, \end{aligned} \quad (3.52)$$

for $i,j \in \{0,1,\dots,n\}$. Similarly, for $\phi = \phi^{NS}$ and n even, we have $n\times n$ $G(w_1,\dots,w_n)$ with entries

$$\begin{aligned} G^{R}_{ij}(w_1,\dots,w_n) &= G^{R}(w_i,w_j) \\ &= \frac{\mathrm{Tr}\left(x^{-2d^{\phi^{R}}}\phi^{R}(w_i)\phi^{R}(w_j)\right)}{\mathrm{Tr}\left(x^{-2d^{\phi^{R}}}\right)}, \\ &= \frac{1}{q+q^{-1}}\sum_{m\in\mathbb{Z}+\frac{1}{2}}\left(\frac{w_j}{w_i}\right)^m \frac{q^{2m}+q^{-2m}}{1+x^{2m}} \end{aligned} \quad (3.53)$$

for $i,j\in\{0,1,\dots,n\}$. For the other case, where n is odd and we have a Ramond fermion, $\bar{G}(w_1,\dots w_n)$ is an $n+1\times n+1$ matrix with entries for $i,j\in\{0,1,\dots,n\}$ given by

$$\bar{G}_{ij}(w_1,\dots,w_n) = \frac{1}{q+q^{-1}}\left(1+\sum_{m\in\mathbb{Z}_{\neq 0}}\left(\frac{w_i}{w_j}\right)^m \frac{q^{2m}+q^{-2m}}{1-x^{2m}}\right), \quad (3.54)$$

for $1 \leq i < j \leq n$ and

$$\bar{G}_{0j} = \frac{\mathrm{Tr}_{\mathcal{F}^{\phi R}}\left(x^{-2d^{\phi R}}\phi_0\right)}{\mathrm{Tr}_{\mathcal{F}^{\phi R}}\left(x^{-2d^{\phi R}}\right)} = \frac{1}{[2]^{\frac{1}{2}}}\frac{(x^2;x^2)_\infty}{(-x^2;x^2)_\infty},$$

for $1 \leq j \leq n$. We have just stated the results here, but a detailed derivation and explicit expressions for the two point functions are given in [60].

3.4.4 Final Expression

We now have the form of the traces over all three components of the Fock spaces identified with our irreducible highest weight modules and can give the general integral expression for the trace (3.34) for each choice of λ_i, $i = 0, 1, 2$. With the insertion of normalisation factors and specialisation of the arguments z_i, this gives the general expression for (3.7) .

For an even number of fermions, we have

$$\begin{aligned}
&\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2)\ldots\Phi_{\varepsilon_n}(z_n)\right)\\
= \;&\delta_{m,s+2t}\prod_{\substack{a,a'\in\mathcal{A}\\ b,b'\in\mathcal{B}\\ i,i'\in\mathcal{I}}}\frac{1}{(q^4;q^4)_\infty}\oint\frac{dv_a}{2\pi i}\oint\frac{dw_a}{2\pi i}\oint\frac{dw_b}{2\pi i}F_0(z_a,v_a,w_a)F_1(z_b,w_b)\\
&\times g_1(z_i,z_{i'})g_1(w_a,w_{a'})g_1(w_b,w_{b'})g_1(v_a,v_{a'})\\
&\times g_{x^-x^-}(w_a,v_a)g_{x^-x^-}(w_a,w_b)g_{x^-x^-}(v_a,w_b)\\
&\times g_{\Phi_2 x^-}(z_i,v_a)g_{\Phi_2 x^-}(z_i,w_a)g_{\Phi_2 x^-}(z_i,w_b)\\
&(v_a w_a w_b)^{\frac{1}{2}}\Theta_{q^{16}}\left(-\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ i\in\mathcal{I}}}\frac{q^{12}z_i^2}{v_a^2w_a^2w_b^2}\right)H(\mathbf{z},\mathbf{v},\mathbf{w})T_{\phi^{NS}}(u_1,\ldots,u_m),
\end{aligned}$$

where $T_\phi(u_i,\ldots,u_m)$ is defined in terms of variables v_a, w_a, w_b through the ordered product relation (3.49) and we specialise to $x = q^2$ in each of the trace contribution expressions. Similarly, in the other two choices of ground state boundary condition,

we have

$$
\begin{aligned}
& \mathrm{Tr}_{V(\lambda_2)}\left(q^{-2\rho}\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2)\dots\Phi_{\varepsilon_n}(z_n)\right) \\
= \quad & -\delta_{m,s+2t}\frac{1}{(q^4;q^4)_\infty}\prod_{\substack{a,a'\in\mathcal{A}\\ b,b'\in\mathcal{B}\\ i,i'\in\mathcal{I}}}\oint\frac{dv_a}{2\pi i}\oint\frac{dw_a}{2\pi i}\oint\frac{dw_b}{2\pi i}F_0(z_a,v_a,w_a)F_1(z_b,w_b) \\
& \quad \times q^4 z_i g_1(z_i,z_{i'})g_1(w_a,w_{a'})g_1(w_b,w_{b'})g_1(v_a,v_{a'}) \\
& \quad \times g_{x^-x^-}(w_a,v_a)g_{x^-x^-}(w_a,w_b)g_{x^-x^-}(v_a,w_b) \\
& \quad \times g_{\Phi_2 x^-}(z_i,v_a)g_{\Phi_2 x^-}(z_i,w_a)g_{\Phi_2 x^-}(z_i,w_b) \\
& \quad (v_a w_a w_b)^{-\frac{1}{2}}\Theta_{q^{16}}\left(-\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ i\in\mathcal{I}}}\frac{q^{20}z_i^2}{v_a^2 w_a^2 w_b^2}\right)H(\mathbf{z},\mathbf{v},\mathbf{w})T_{\phi^{NS}}(u_1,\dots,u_m),
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2)\dots\Phi_{\varepsilon_n}(z_n)\right) \\
= \quad & \delta_{m,s+2t}\frac{1}{(q^4;q^4)_\infty}\prod_{\substack{a,a'\in\mathcal{A}\\ b,b'\in\mathcal{B}\\ i,i'\in\mathcal{I}}}\oint\frac{dv_a}{2\pi i}\oint\frac{dw_a}{2\pi i}\oint\frac{dw_b}{2\pi i}F_0(z_a,v_a,w_a)F_1(z_b,w_b) \\
& \quad \times(-q^4 z_i)^{\frac{1}{2}}g_1(z_i,z_{i'})g_1(w_a,w_{a'})g_1(w_b,w_{b'})g_1(v_a,v_{a'}) \\
& \quad \times g_{x^-x^-}(w_a,v_a)g_{x^-x^-}(w_a,w_b)g_{x^-x^-}(v_a,w_b) \\
& \quad \times g_{\Phi_2 x^-}(z_i,v_a)g_{\Phi_2 x^-}(z_i,w_a)g_{\Phi_2 x^-}(z_i,w_b) \\
& \quad \Theta_{q^4}\left(\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ i\in\mathcal{I}}}\frac{q^6 z_i}{v_a w_a w_b}\right)H(\mathbf{z},\mathbf{v},\mathbf{w})T_{\phi^R}(u_1,\dots,u_m).
\end{aligned}
$$

For an odd number of fermion fields, i.e. odd s, the trace over the Neveu-Schwarz fermion field is zero and so we only obtain non-zero trace for the choice $V(\lambda_i) = V(\lambda_1)$. This is given by the same expression as above, with the parity of the fermion number manifesting itself through the definition of T_ϕ.

3.5 Bosonic Trace for m-particle Form-Factors

Whilst the vertex operator approach in the one boson, one fermion scheme has been applied to the computation of n-point correlation functions previously (see [50], [59], [60]), the inclusion of type II vertex operators in the trace expressions, with a view to computing form-factors, has not been considered. In this section, we consider the bosonic and lattice contributions to the m-particle form-factors.

The computation works in much the same way as when considering the n-point function (3.34), but now we introduce type II vertex operators and so will require their normal ordering relations and trace contributions. We compute the trace

$$\mathrm{Tr}_{V(\lambda_i)}\left(x^{-\rho}\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2)\dots\Phi_{\varepsilon_n}(z_n)\Psi_{\ell_1}(\xi_1)\Psi_{\ell_2}(\xi_2)\dots\Psi_{\ell_m}(\xi_m)\right), \tag{3.55}$$

which, with specialisation of arguments and the correct pre-factors (arising through normal ordering, normalisation and the relation between dual type II vertex operators and the usual Ψ_ℓ operators), will give us an expression for (3.8).

For the normal ordering of the type I vertex operators, we again use index notation (3.38) and can use the result (3.42) to immediately write

$$\begin{aligned}
&\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2)\dots\Phi_{\varepsilon_n}(z_n)\Psi_{\ell_1}(\xi_1)\Psi_{\ell_2}(\xi_2)\dots\Psi_{\ell_m}(\xi_m)\\
=&\prod_a\oint\frac{dv_a}{2\pi i}\oint\frac{dw_a}{2\pi i}F_1(z_a,v_a,w_a)\prod_b\oint\frac{dw_b}{2\pi i}F_0(z_b,w_b)H(\mathbf{z},\mathbf{v},\mathbf{w})\\
&\times\prod_{\substack{a\in\mathcal{A}\\ b\in\mathcal{B}\\ i\in\mathcal{I}}}:\Phi_2(z_i)\hat{x}^-(w_a)\hat{x}^-(v_a)\hat{x}^-(w_b):\Psi_{\ell_1}(\xi_1)\Psi_{\ell_2}(\xi_2)\dots\Psi_{\ell_m}(\xi_m).
\end{aligned}$$

We now introduce index sets

$$\begin{aligned}
\mathcal{D} &= \{d; 1\le d\le m, \ell_j=0\}\\
\mathcal{E} &= \{e; 1\le e\le m, \ell_e=1\},\\
\mathcal{J} &= \{j; 1\le j\le m\}.
\end{aligned} \tag{3.56}$$

We can write the more complicated type II vertex operator component as

$$\Psi_1(z) \quad = \quad \oint \frac{du}{2\pi i} G_1(\xi,u) : \Psi_0(\xi)x^+(u) : \Omega(q^{-2}\xi)\phi(u),$$

where

$$G_1(z,w) \quad = \quad (q-q^{-1})(-\xi w^3)^{-\frac{1}{2}} \frac{(q^3 w/z;q^4)_\infty}{(w/q^3 z;q^4)_\infty}, \qquad |\frac{w}{q^3 z}| < 1.$$

This means that the bosonic type II contribution to the trace is built from ordered operators of the form

$$\begin{aligned}
\mathcal{O}_D(\xi_d) &= \hat{\Psi}_0(z_d),\ d \in \mathcal{D}, \\
\mathcal{O}_E(\xi_e, u_e) &= \ :\hat{\Psi}_0(z_e)\hat{x}^+(u_e):,\ \ e \in \mathcal{E}.
\end{aligned}$$

We know that our type I vertex operators always lie to the left of our type vertex operators and so (using the normal ordering relations appearing in Appendix C) for each $a \in \mathcal{A}$ and $d \in \mathcal{D}$, we have a contribution

$$\begin{aligned}
\mathcal{O}_A(z_a,v_a,w_a)\mathcal{O}_D(\xi_d) &= \mathcal{N}_{AD}(z_a,v_a,w_a;\xi_d) : \mathcal{O}_A(z_a,v_a,w_a)\mathcal{O}_D(\xi_d) : \\
&= \left(q\xi_d/z_a;q^4\right)_\infty \left(\xi_d/qz_a;q^4\right)_\infty (-q^4 z_a)^{-1/2} \\
&\quad \times \frac{(q^3\xi_d/v_a;q^4)_\infty}{(q^5\xi_d/v_a;q^4)_\infty} v_a^{1/2} \frac{(q^3\xi_d/w_a;q^4)_\infty}{(q^5\xi_d/w_a;q^4)_\infty} w_a^{1/2} \\
&\quad \times : \mathcal{O}_A(z_a,v_a,w_a)\mathcal{O}_D(\xi_d) : . \qquad (3.57)
\end{aligned}$$

Similarly, for each $a \in \mathcal{A}$ and $e \in \mathcal{E}$, we have a contribution

$$\begin{aligned}
\mathcal{O}_A(z_a,v_a,w_a)\mathcal{O}_E(\xi_e,u_e) &= \mathcal{N}_{AE}(z_a,v_a,w_a;\xi_e,u_e) : \mathcal{O}_A\mathcal{O}_E : \\
&= \left(q\xi_e/z_a;q^4\right)_\infty \left(\xi_e/qz_a;q^4\right)_\infty (-q^4 z_a)^{-1/2} \\
&\quad \times \frac{(q^3\xi_e/v_a;q^4)_\infty}{(q^5\xi_e/v_a;q^4)_\infty} v_a^{1/2} \frac{(q^3\xi_e/w_a;q^4)_\infty}{(q^5\xi_e/w_a;q^4)_\infty} w_a^{1/2} \\
&\quad \times \frac{(-q^4 z_a)(1-u_e/q^4 z_a)}{v_a w_a (1-u_e/v_a)(1-u_e/w_a)} \\
&\quad \times : \mathcal{O}_A(z_a,v_a,w_a)\mathcal{O}_E(\xi_e,u_e) : . \qquad (3.58)
\end{aligned}$$

We continue in this way to compute normal ordering factors for each possible pairing $\Phi_{\varepsilon_i}(z_i)\Psi_{\ell_j}(\xi_j)$.

In the same way as in the normal ordering calculation for type I vertex operators when computing the n-point function, when normal ordering our type II vertex operators, we need to keep track of the order in which they appear. For each $d \in \mathcal{D}$ and $e \in \mathcal{E}$ such that $d < e$, we have a contribution

$$\begin{aligned} \mathcal{O}_D(\xi_d)\mathcal{O}_E(\xi_d, u_e) &= \mathcal{N}_{DE}(\xi_d; \xi_e, u_e) : \mathcal{O}_D(\xi_d)\mathcal{O}_E(\xi_d, u_e) : \\ &= \frac{(q^4\xi_e/\xi_d; q^4, q^4)_\infty \, (\xi_e/\xi_d; q^4, q^4)_\infty}{(q^2\xi_e/\xi_d; q^4, q^4)^2_\infty}(-q^2\xi_d)^{-\frac{1}{4}}\frac{(u_e/q\xi_d; q^4)_\infty}{(u_e/q^3\xi_d; q^4)_\infty} \\ &\quad \times : \mathcal{O}_D(\xi_d)\mathcal{O}_E(\xi_d, u_e) : . \end{aligned} \tag{3.59}$$

For each $d \in \mathcal{D}$ and $e \in \mathcal{E}$ such that $e < d$, we have

$$\begin{aligned} \mathcal{O}_E(\xi_d, u_e)\mathcal{O}_D(\xi_d) &= \mathcal{N}_{ED}(\xi_e, u_e; \xi_d) : \mathcal{O}_D(\xi_d)\mathcal{O}_E(\xi_d, u_e) : \\ &= \frac{(q^4\xi_d/\xi_e; q^4, q^4)_\infty \, (\xi_d/\xi_e; q^4, q^4)_\infty}{(q^2\xi_d/\xi_e; q^4, q^4)^2_\infty}(-q^2\xi_e)^{1/4}\frac{(q^3\xi_d/u_e; q^4)_\infty}{(q\xi_d/u_e; q^4)_\infty}u_e^{-\frac{1}{2}} \\ &\quad \times : \mathcal{O}_D(\xi_d)\mathcal{O}_E(\xi_d, u_e) : . \end{aligned} \tag{3.60}$$

Next, for $d, d' \in \mathcal{D}$ with $d < d'$, we have a contribution

$$\begin{aligned} \mathcal{O}_D(\xi_d)\mathcal{O}_D(\xi_{d'}) &= \mathcal{N}_{DD}(\xi_d; \xi_{d'}) : \mathcal{O}_D(\xi_d)\mathcal{O}_D(\xi_{d'}) : \\ &= \frac{(q^4\xi_{d'}/\xi_d; q^4, q^4)_\infty \, (\xi_{d'}/\xi_d; q^4, q^4)_\infty}{(q^2\xi_{d'}/\xi_d; q^4, q^4)^2_\infty}(-q^2\xi_d)^{1/4} \\ &\quad \times : \mathcal{O}_D(\xi_d)\mathcal{O}_D(\xi_{d'}) : . \end{aligned} \tag{3.61}$$

Finally, for $e, e' \in \mathcal{E}$ with $e < e'$, we have

$$\begin{aligned} \mathcal{O}_E(\xi_e)\mathcal{O}_E(\xi_{e'}) &= \mathcal{N}_{EE}(\xi_e; \xi_{e'}) : \mathcal{O}_E(\xi_e)\mathcal{O}_E(\xi_{e'}) : \\ &= \frac{(q^4\xi_{e'}/\xi_e; q^4, q^4)_\infty \, (\xi_{e'}/\xi_e; q^4, q^4)_\infty}{(q^2\xi_{e'}/\xi_e; q^4, q^4)^2_\infty}(-q^2\xi_e)^{-\frac{1}{4}} \\ &\quad \times u_e^{\frac{1}{2}}(1 - q^{-2}u_{e'}/u_e)\frac{(q^3\xi_{e'}/u_e; q^4)_\infty}{(q\xi_{e'}/u_e; q^4)_\infty}\frac{(u_{e'}/q\xi_e; q^4)_\infty}{(u_{e'}/q^3\xi_e; q^4)_\infty} \\ &\quad \times : \mathcal{O}_E(\xi_e)\mathcal{O}_E(\xi_{e'}) : . \end{aligned} \tag{3.62}$$

A full list of normal ordering factors is given in Appendix C. Bringing all of this together, we have the following expression for the normal ordering of a string of type I and type II vertex operators.

$$\begin{aligned}
& \hat{\Phi}_{\varepsilon_1}(z_1)\hat{\Phi}_{\varepsilon_2}(z_2)\ldots\hat{\Phi}_{\varepsilon_n}(z_n)\hat{\Psi}_{\ell_1}(\xi_1)\hat{\Psi}_{\ell_2}(\xi_2)\ldots\hat{\Psi}_{\ell_m}(\xi_m) \\
= & \prod_{a,b,c,d,e} \oint \frac{dv_a}{2\pi i} \oint \frac{dw_a}{2\pi i} F_0(z_a, v_a, w_a) \oint \frac{dw_b}{2\pi i} F_1(z_b, w_b) \oint \frac{du_e}{2\pi i} G_1(\xi_e, u_e) \\
& \times \mathcal{N}_{AD}(z_a, v_a, w_a; \xi_d)\mathcal{N}_{AE}(z_a, v_a, w_a; \xi_e, u_e) \\
& \times \mathcal{N}_{BD}(z_b, w_b; \xi_d)\mathcal{N}_{BE}(z_b, w_b; \xi_e, u_e)\mathcal{N}_{CD}(z_c; \xi_d)\mathcal{N}_{CE}(z_c; \xi_e, u_e) \\
& \times \prod_{d<e} \mathcal{N}_{DE}(\xi_d; \xi_e, u_e) \prod_{e<d} \mathcal{N}_{ED}(\xi_e, u_e; \xi_d) \\
& \times \prod_{d<d'} \mathcal{N}_{DD}(\xi_d; \xi_{d'}) \prod_{e<e'} \mathcal{N}_{EE}(\xi_e, u_e; \xi_{e'}, u_{e'}) H(\mathbf{z}, \mathbf{v}, \mathbf{w}) \\
& \times \prod_{a,b,e,i,j} : \hat{\Phi}(z_i)\hat{\Psi}(z_j)\hat{x}^-(v_a)\hat{x}^-(w_a)\hat{x}^-(w_b)\hat{x}^+(u_e) : ,
\end{aligned}$$

where $a \in \mathcal{A}, b \in \mathcal{B}, e \in \mathcal{E},\ i \in \mathcal{I}$ and $j \in \mathcal{J}$. We now use (3.35) along with the trace contributions detailed in Appendix C to compute the trace

$$\mathrm{Tr}_{V(\Lambda_i)}\left(x^{-\rho} \prod_{a,b,e,i,j} : \hat{\Phi}(z_i)\hat{\Psi}(z_j)\hat{x}^-(v_a)\hat{x}^-(w_a)\hat{x}^-(w_b)\hat{x}^+(u_e) : \right).$$

We can combine self contributions and contributions from two of the same type of operator as in the previous section for $\Phi_2(z)$ and $x^-(w)$ (3.47). For $\Psi_0(\xi)$, we have

$$g_{\Psi_0} g_{\Psi_0} g_{\Psi_0\Psi_0}(\xi_1, \xi_2) = \prod_{j,j'=1}^{2} \frac{(x^2 q^4 \xi_{j'}/\xi_j; q^4, q^4, x^2)_\infty (x^2 \xi_{j'}/\xi_j; q^4, q^4, x^2)_\infty}{(q^2 x^2 \xi_{j'}/\xi_j; q^4, q^4, x^2)_\infty^2},$$

and so we define

$$g_2(\xi_j, \xi_{j'}) = \frac{(q^4 x^2 \xi_{j'}/\xi_j; q^4, q^4, x^2)_\infty (x^2 \xi_{j'}/\xi_j; q^4, q^4, x^2)_\infty}{(x^2 q^2 \xi_{j'}/\xi_j; q^4, q^4, x^2)_\infty^2}. \tag{3.63}$$

Similarly, for $x^+(w)$, we have

$$g_{x^+}g_{x^+}g_{x^+x^+}(u_1,u_2) = \prod_{e,e'=1}^{2}(x^2q^{-2}u_e/u_{e'};x^2)_\infty,$$

and we define

$$g_3(u_e,u_{e'}) = (x^2q^{-2}u_e/u_{e'};x^2)_\infty. \tag{3.64}$$

All of these components come together to give the following expression for the trace.

$$\begin{aligned}
&\mathrm{Tr}_{V(\Lambda_i)}\left(\prod_{a,b,e,i,j} :\hat{\Phi}_2(z_i)\hat{\Psi}_0(z_j)\hat{x}^-(v_a)\hat{x}^-(w_a)\hat{x}^-(w_b)\hat{x}^+(u_e):\right)\\
&=\frac{1}{(q^4;q^4)_\infty}\prod_{a,a'\in\mathcal{A}}\prod_{b,b'\in\mathcal{B}}\prod_{e,e'\in\mathcal{E}}\prod_{i,i'\in\mathcal{I}}\prod_{j,j'\in\mathcal{J}}\\
&\times g_1(z_i,z_{i'})g_1(w_a,w_{a'})g_1(w_b,w_{b'})g_1(v_a,v_{a'})g_2(\xi_j,\xi_{j'})g_3(u_e,u_{e'})\\
&\times g_{x^-x^-}(w_a,v_a)g_{x^-x^-}(w_a,w_b)g_{x^-x^-}(v_a,w_b)\\
&\times g_{\Phi_2x^-}(z_i,v_a)g_{\Phi_2x^-}(z_i,w_a)g_{\Phi_2x^-}(z_i,w_b)\\
&\times g_{x^-x^+}(v_a,u_e)g_{x^-x^+}(w_a,u_e)g_{x^-x^+}(w_b,u_e)\\
&\times g_{\Psi_0x^-}(\xi_j,v_a)g_{\Psi_0x^-}(\xi_j,w_a)g_{\Psi_0x^-}(\xi_j,w_b)\\
&\times g_{\Phi_2\Psi_0}(z_i,\xi_j)g_{\Phi_2x^+}(z_i,u_e)g_{\Psi_0x^+}(\xi_j,u_e)\\
&\times\mathrm{Tr}_{V(\lambda_i)}\left(x^{-\rho}\prod_{a,b,e,i,j}(v_aw_aw_bu_e)^{\frac{1}{2}}\left(\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{\partial}{4}}\right),
\end{aligned}$$

where the trace over the lattice space is dependent on the choice of λ_i and is computed according to Section 3.4.2.

If we now bring in the normal ordering contribution, the final expression for the bosonic contributions to the form-factor trace (3.55) is given by

$$\begin{aligned}
&\mathrm{Tr}_{V(\lambda_i)}\left(x^{-\rho}\hat{\Phi}_{\varepsilon_1}(z_1)\hat{\Phi}_{\varepsilon_2}(z_2)\dots\hat{\Phi}_{\varepsilon_n}(z_n)\hat{\Psi}_{\ell_1}(\xi_1)\hat{\Psi}_{\ell_2}(\xi_2)\dots\hat{\Psi}_{\ell_m}(\xi_m)\right)\\
= &\prod_{a,b,c,d,e}\oint\frac{dv_a}{2\pi i}\oint\frac{dw_a}{2\pi i}F_0(z_a,v_a,w_a)\oint\frac{dw_b}{2\pi i}F_1(z_b,w_b)\oint\frac{du_e}{2\pi i}G_1(\xi_e,u_e)\\
&\times\frac{1}{(q^4;q^4)_\infty}\mathcal{N}(\mathbf{z},\boldsymbol{\xi},\mathbf{v},\mathbf{w},\mathbf{u})G(\mathbf{z},\boldsymbol{\xi},\mathbf{v},\mathbf{w},\mathbf{u})\\
&\times\mathrm{Tr}_{V(\lambda_i)}\left(x^{-\rho}\prod_{a,b,e,i,j}(v_a w_a w_b u_e)^{\frac{1}{2}}\left(\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{\partial}{4}}\right),
\end{aligned}\tag{3.65}$$

where we define

$$\begin{aligned}
G(\mathbf{z},\boldsymbol{\xi},\mathbf{v},\mathbf{w},\mathbf{u}) \;=\; &\prod_{a,a'\in\mathcal{A}}\;\prod_{b,b'\in\mathcal{B}}\;\prod_{e,e'\in\mathcal{E}}\;\prod_{i,i'\in\mathcal{I}}\;\prod_{j,j'\in\mathcal{J}}\\
&\times g_1(z_i,z_{i'})g_1(w_a,w_{a'})g_1(w_b,w_{b'})g_1(v_a,v_{a'})g_2(\xi_j,\xi_{j'})g_3(u_e,u_{e'})\\
&\times g_{x^-x^-}(w_a,v_a)g_{x^-x^-}(w_a,w_b)g_{x^-x^-}(v_a,w_b)\\
&\times g_{\Phi_2x^-}(z_i,v_a)g_{\Phi_2x^-}(z_i,w_a)g_{\Phi_2x^-}(z_i,w_b)\\
&\times g_{x^-x^+}(v_a,u_e)g_{x^-x^+}(w_a,u_e)g_{x^-x^+}(w_b,u_e)\\
&\times g_{\Psi_0x^-}(\xi_j,v_a)g_{\Psi_0x^-}(\xi_j,w_a)g_{\Psi_0x^-}(\xi_j,w_b)\\
&\times g_{\Phi_2\Psi_0}(z_i,\xi_j)g_{\Phi_2x^+}(z_i,u_e)g_{\Psi_0x^+}(\xi_j,u_e),
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}(\mathbf{z},\boldsymbol{\xi},\mathbf{v},\mathbf{w},\mathbf{u}) &= \prod_{a,b,e,i,j} \mathcal{N}_{AD}(z_a,v_a,w_a;\xi_d)\mathcal{N}_{AE}(z_a,v_a,w_a;\xi_e,u_e)\\
&\times\mathcal{N}_{BD}(z_b,w_b;\xi_d)\mathcal{N}_{BE}(z_b,w_b;\xi_e,u_e)\mathcal{N}_{CD}(z_c;\xi_d)\mathcal{N}_{CE}(z_c;\xi_e,u_e)\\
&\times\prod_{a<a'}\mathcal{N}_{AA}(z_a,v_a,w_a;z_{a'},v_{a'},w_{a'})\prod_{b<b'}\mathcal{N}_{BB}(z_b,w_b;z_{b'},w_{b'})\\
&\times\prod_{c<c'}\mathcal{N}_{CC}(z_c;z_{c'})\prod_{c<b}\mathcal{N}_{CB}(z_c;z_b,w_b)\prod_{c>b}\mathcal{N}_{BC}(z_b,w_b;z_c)\\
&\times\prod_{c<a}\mathcal{N}_{CA}(z_c;z_a,v_a,w_a)\prod_{a<c}\mathcal{N}_{AC}(z_a,v_a,w_a;z_c)\\
&\times\prod_{a<b}\mathcal{N}_{AB}(z_a,v_a,w_a;z_b,w_b)\prod_{b<a}\mathcal{N}_{BA}(z_b,w_b;z_a,v_a,w_a)\\
&\times\prod_{d<e}\mathcal{N}_{DE}(\xi_d;\xi_e,u_e)\prod_{e<d}\mathcal{N}_{ED}(\xi_e,u_e;\xi_d)\\
&\times\prod_{d<d'}\mathcal{N}_{DD}(\xi_d;\xi_{d'})\prod_{e<e'}\mathcal{N}_{EE}(\xi_e,u_e;\xi_{e'},u_{e'}).
\end{aligned}$$

3.5.1 Zero-mode Contributions

For each λ_i, we now give the general form of the zero-mode contribution to the trace using the relations from Section (3.4.2). We recall the free field realisations (3.15) and (3.16) for highest weight modules $V(2\Lambda_0)$ and $V(2\Lambda_1)$. The lattice space over which we take the trace depends on the number or fermion fields appearing in the trace. For $|\mathcal{B}|+|\mathcal{E}|$ odd, we have an odd number of fermions $\phi(w)$ and so the contribution from taking a trace over $V(2\Lambda_0)$ is

$$\begin{aligned}
&\mathrm{Tr}_{e^\alpha\mathbb{C}[2Q]}\left(q^{-2\rho}\left(\prod_{a,b,e,i,j}\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{\partial}{4}}\right)\\
&=\left(\prod_{a,b,e,i,j}\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{1}{2}}\Theta_{q^{16}}\left(-\prod_{a,b,e,i,j}\frac{q^{18}u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right),
\end{aligned}$$

whereas the lattice space trace over $V(2\Lambda_1)$ is

$$\begin{aligned}
&\mathrm{Tr}_{\mathbb{C}[2Q]}\left(q^{-2\rho}\left(\prod_{a,b,e,i,j}\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{\partial}{4}}\right)\\
&=\Theta_{q^{16}}\left(-\prod_{a,b,e,i,j}\frac{q^{10}u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right).
\end{aligned}$$

For $|\mathcal{B}| + |\mathcal{E}|$ even, we have an even number of fermions $\phi(w)$ appearing and the previous contributions are swapped. So, for the trace over $V(2\Lambda_0)$, we have a contribution

$$\begin{aligned}
&\mathrm{Tr}_{\mathbb{C}[2Q]}\left(q^{-2\rho}\prod_{a,b,e,i,j}\left(\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{\partial}{4}}\right)\\
&= \Theta_{q^{16}}\left(-\prod_{a,b,e,i,j}\frac{q^{10}u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right),
\end{aligned}$$

whilst from the trace over $V(2\Lambda_1)$, we have

$$\begin{aligned}
&\mathrm{Tr}_{e^{\alpha}\mathbb{C}[2Q]}\left(q^{-2\rho}\prod_{a,b,e,i,j}\left(\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{\partial}{4}}\right)\\
&= \left(\prod_{a,b,e,i,j}\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{1}{2}}\Theta_{q^{16}}\left(-\prod_{a,b,e,i,j}\frac{q^{18}u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right).
\end{aligned}$$

Finally, for $V(\Lambda_0 + \Lambda_1)$, we have the following contribution for any number of fermions

$$\begin{aligned}
&\mathrm{Tr}_{e^{\frac{\alpha}{2}}\mathbb{C}[Q]}\left(q^{-2\rho}\prod_{a,b,e,i,j}\left(\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{\partial}{4}}\right)\\
&= \left(\prod_{a,b,e,i,j}\frac{q^6u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{1}{4}}\Theta_{q^4}\left(-\prod_{a,b,e,i,j}\left(\frac{q^{10}u_e^2z_i^2}{v_a^2w_a^2w_b^2\xi_j}\right)^{\frac{1}{2}}\right).
\end{aligned}$$

3.6 The Shiraishi Realisation for Fermions

The approach to computing form-factors using the one boson, one fermion free field realisation has so far given us a reasonable integral expression (3.65) for the bosonic and lattice contributions. The problem of computing the fermionic contribution to the trace proves to be a difficult one. We no longer only encounter fermion fields $\phi(w)$ as in the case of the n-point function, which we could deal with by using Pfaffians (3.51), as in [60]. In this case, we will be taking the trace over an ordered product of a mixture of both fermions and fermion emission operators (3.27).

Moving forward, this section introduces a novel approach for dealing with fermionic traces of the type that appear. The approach hinges on consideration of the representation theory of the q-deformed Virasoro algebra appearing in [78]. The paper investigates free field realisations for the elliptic quantum group $\mathcal{A}_{q,p}(\widehat{sl}_2)$ based on this representation theory and shows that, at a certain value of p, the vertex operators of the elliptic algebra are related to the vertex operators of the Ising model [22]. Crucially, these Ising vertex operators can be identified with our fermion emission operators and fermionic fields as in [22] and [61]. This means that we are able to exploit a scheme, introduced by Shiraishi in [78], whereby the vertex operators of the Ising model have free field realisation in terms of operators built from the q-Virasoro generators. More detail on the q-Virasoro algebra can be found in [95], [96] and [97].

Following the conventions of [78], we introduce operators $\chi(\zeta)$ and $\Lambda_{\pm}(\zeta)$. These are defined in terms of q-Virasoro oscillators λ_n as

$$\begin{aligned}
\chi(\zeta) &= :\exp\left\{-\sum_{n\neq 0}\frac{(-1)^{-n/2}q^n\lambda_n\zeta^{-n}}{(1-(-q^2)^n)}\right\}: e^{\frac{Q_\lambda}{4}}\zeta^{\frac{\lambda_0}{4}}\zeta^{\frac{1}{8}} \\
&= \exp\left\{-\sum_{n>0}\frac{(-1)^{n/2}q^{-n}\lambda_{-n}\zeta^{n}}{(1-(-q^2)^{-n})}\right\}\exp\left\{-\sum_{n>0}\frac{(-1)^{-n/2}q^{n}\lambda_{n}\zeta^{-n}}{(1-(-q^2)^{n})}\right\} \\
&\quad \cdot e^{\frac{Q_\lambda}{4}}\zeta^{\frac{\lambda_0}{4}}\zeta^{\frac{1}{8}} \\
\Lambda_{\pm}(\zeta) &= :\exp\left\{\pm\sum_{n\neq 0}\lambda_n\zeta^{-n}\right\}: i^{\pm 1}q^{\mp\frac{\lambda_0}{2}},
\end{aligned}$$

where the λ_n satisfy

$$[\lambda_n,\lambda_m] = -\delta_{m+n,0}\frac{1}{n}\frac{(1-q^{-n})(1-(-q^2)^n)}{(1+(-q)^n)},$$

and for the zero mode parts, we have

$$[\lambda_0, Q_\lambda] = 4.$$

We also introduce the q-Virasoro current $T(z)$, which has free field realisation[2]

$$T(\zeta) = \Lambda_+(iq^{-\frac{1}{2}}\zeta) + \Lambda_-(-iq^{\frac{1}{2}}\zeta). \tag{3.66}$$

We define associated Fock spaces $\mathcal{F}_r$ by

$$\mathcal{F}_r := \mathbb{C}[\lambda_{-1}, \lambda_{-2}, \ldots]e^{\frac{rQ}{4}}\left|0\right\rangle, \quad r \in \mathbb{Z},$$

with $\lambda_n \left|0\right\rangle = 0$, $n \geq 0$. The grading operator in the Shiraishi realisation is denoted by $\hat{\rho}$ and acts as

$$\begin{aligned} \hat{\rho}\lambda_{-n} \exp\left\{\frac{rQ_\lambda}{4}\right\}\left|0\right\rangle &= \left(-n - \frac{\lambda_0^2}{8}\right)\lambda_{-n}\exp\left\{\frac{rQ_\lambda}{4}\right\}\left|0\right\rangle \\ &= \left(-n - \frac{r^2}{8}\right)\lambda_{-n}\exp\left\{\frac{rQ_\lambda}{4}\right\}\left|0\right\rangle. \end{aligned}$$

The Fock spaces $\mathcal{F}_r$ are reducible because of the existence of singular vectors. In order to resolve this, we introduce BRST operator $\mathcal{Q}$ and obtain the irreducible part as the zero-th cohomology group in a complex of graded Fock modules [78, 86, 92, 95–97]. More detail on this method will be given in Chapter 5 in the setting of the q-Wakimoto free field realisation, but we will now give an overview of its application in this setting.

The BRST operator $\mathcal{Q}$ satisfies two important properties:

1. $\mathcal{Q}$ is nilpotent, satisfying $\mathcal{Q}^2 = 0$.
2. $\mathcal{Q}$ commutes with the q-Virasoro current $T(z)$.

The general idea is to construct the singular vectors of the Fock space by acting on highest weight vectors with $\mathcal{Q}$. Taking our lead from Shiraishi, we define our BRST operator $\mathcal{Q}$ to be

$$\mathcal{Q} = \oint \frac{dz}{2\pi iz} S_+(z),$$

[2]The q-Virasoro generators λ_n are related to those of [95] by $\lambda_n = -h_n$ and $\lambda_0 = -2\beta a_0$. Our $\Lambda_\pm$ are related to the $\Lambda^\pm$ by $\Lambda^+(z) = \Lambda_-(iq^{-\frac{1}{2}}z)$, $\Lambda^-(z) = \Lambda_-(-iq^{\frac{1}{2}})$.

where $S_+(z))$ is the so-called screening current

$$S_+(z) =: \exp\left(-\sum_{n\neq 0}\frac{1+(-q)^{-n}}{1-q^n}(-1)^{-\frac{n}{2}}q^n\lambda_n z^{-n}\right) : e^{Q_\lambda}z^{\lambda_0}z^2.$$

The properties of $\mathcal{Q}$ mean that we have the complex

$$\ldots \xrightarrow{\mathcal{Q}} \mathcal{F}_{r-4} \xrightarrow{\mathcal{Q}} \mathcal{F}_r \xrightarrow{\mathcal{Q}} \mathcal{F}_{r+4} \xrightarrow{\mathcal{Q}} \ldots ,$$

where $\mathrm{Im}_{\mathcal{F}_{r-4}}(\mathcal{Q}) \subset \mathrm{Ker}_{\mathcal{F}_r}(\mathcal{Q})$. The nilpotency of $\mathcal{Q}$ means that the physical states of the system (comprising the irreducible part of the Fock module that we desire) lie in the kernel of $\mathcal{Q}$. The redundant singular vectors, on the otherhand, lie in the image of $\mathcal{Q}$ and need to be removed. The irreducible part is then realised by the zero-th cohomology group

$$H_r^0 = \mathrm{Ker}_{\mathcal{F}_r}(\mathcal{Q}) \setminus \mathrm{Im}_{\mathcal{F}_{r-4}}(\mathcal{Q}),$$

whilst all other cohomologies of the complex vanish. The trace over the irreducible part of the Fock space $\mathcal{F}_r$ obtained in this way is then given by an alternating sum over traces of graded operators over graded Fock spaces,

$$\mathrm{Tr}_{H_r^0}(\mathcal{O}) = \sum_{s\in\mathbb{Z}}(-1)^s \mathrm{Tr}_{\mathcal{F}_r^{[s]}}\left(\mathcal{O}^{[s]}\right), \tag{3.67}$$

where

$$\mathcal{Q}\mathcal{O}^{[s-1]} = \mathcal{O}^{[s]}\mathcal{Q}, \quad \mathcal{O}^{[0]} = \mathcal{O}, \tag{3.68}$$

and $\mathcal{F}_r^{[s]} = \mathcal{F}_{r-4s}$.

3.6.1 Characters

We can now use the above to compute the character of the irreducible parts of the Fock spaces. Assuming that $[\mathcal{Q}, x^{-\hat{\rho}}] = 0$, then we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{F}_r}\left(x^{-\hat{\rho}}\right) &= x^{\frac{r^2}{8}}\prod_{n=1}^{\infty}\frac{1}{(1-x^n)}\\ &= \frac{x^{\frac{r^2}{8}}}{(x;x)_\infty},\end{aligned}$$

where we have used a modification of the bosonic trace formula (3.35). Using (3.67), we then find that

$$\mathrm{Tr}_{H_r^0}\left(x^{-\hat{\rho}}\right) = \frac{x^{\frac{r^2}{8}}}{(x;x)_\infty}\Theta_{x^4}\left(x^{r+2}\right).$$

For $r = 1$ and $r = 0$, this gives us

$$\mathrm{Tr}_{H_1^0}\left(x^{-\hat{\rho}}\right) = x^{\frac{1}{8}}(-x^2;x^2)_\infty,$$

and

$$\mathrm{Tr}_{H_0^0}\left(x^{-\hat{\rho}}\right) = (-x;x^2)_\infty,$$

where we have used theta function and q-infinite product identities listed in Appendix A. We note that these coincide with the characters of the Ramond and Neveu-Schwarz fermion Fock spaces [59, 60], motivating the identifications that follow.

3.6.2 Identifications

The Ramond and Neveu-Schwarz fermion Fock spaces are identified with the irreducible modules obtained for $r = 0$ and $r = 1$, respectively:

$$\mathcal{F}^R = H_1^0, \quad \mathcal{F}^{NS} = H_0^0.$$

We now define a second screening current $S_-(z)$ by

$$\begin{aligned} S_-(z) &= :\exp\left\{\sum_{n\neq 0}\frac{(1+(-q)^n)}{(1-(-q^2)^n)}q^{\frac{n}{2}}\lambda_n z^{-n}\right\}: e^{-\frac{Q_\lambda}{2}}z^{-\frac{\lambda_0}{2}}z^{\frac{1}{2}} \\ &= \exp\left\{\sum_{n>0}\frac{(1+(-q)^{-n})}{(1-(-q^2)^{-n})}q^{-\frac{n}{2}}\lambda_{-n}z^{n}\right\}\exp\left\{\sum_{n>0}\frac{(1+(-q)^n)}{(1-(-q^2)^n)}q^{\frac{n}{2}}\lambda_n z^{-n}\right\} \\ &\quad \times e^{-\frac{Q_\lambda}{2}}z^{-\frac{\lambda_0}{2}}z^{\frac{1}{2}}. \end{aligned}$$

With this, we introduce fermion fields $\psi^R(w)$, $\psi^{NS}(w)$ and Ising vertex operators Φ^R_{NS}, Φ^{NS}_R in terms of q-Virasoro objects by

$$\psi^{NS}(\zeta) = \alpha\left(\Lambda_+(iq^{-\frac{1}{2}}\zeta)+\Lambda_-(-iq^{\frac{1}{2}}\zeta)\right), \quad \text{on} \quad H^0_0 \tag{3.69}$$

$$\psi^{R}(\zeta) = \alpha\left(\Lambda_+(iq^{-\frac{1}{2}}\zeta)+\Lambda_-(-iq^{\frac{1}{2}}\zeta)\right), \quad \text{on} \quad H^0_1, \tag{3.70}$$

and

$$\Phi^R_{NS}(\zeta) = \chi(\zeta) \tag{3.71}$$

$$\Phi^{NS}_R(\zeta) = \beta\oint\frac{dz}{2\pi i}\frac{1}{z}\chi(\zeta)S_-(z)\frac{\Theta_{-q^2}(iq^{\frac{1}{2}}\zeta/z)}{\Theta_{-q^2}(-iq^{\frac{1}{2}}\zeta/z)}, \tag{3.72}$$

as in [78] and [97]. In the above, α and β are the following q-dependent factors

$$\begin{aligned} \alpha &= i(q^{\frac{1}{2}}-q^{-\frac{1}{2}}) \\ \beta &= \frac{(-q,-q^2)_\infty(-q^2;-q^2)_\infty}{(q;-q^2)_\infty(q^2;-q^2)_\infty}. \end{aligned}$$

Using the normal ordering relations from Appendix D, we can compute the exchange relation between fermion emission operators and fermions. We set $\zeta = \zeta_1/\zeta_2$ and compute

$$\frac{\mathcal{N}_{\chi\Lambda_+}(\zeta_1,iq^{-\frac{1}{2}}\zeta_2)}{\mathcal{N}_{\Lambda_+\chi}(iq^{-\frac{1}{2}}\zeta_2,\zeta_1)} = -\zeta^{-1}\frac{(q\zeta^2;q^4)_\infty(q^3\zeta^{-2};q^4)_\infty}{(q\zeta^{-2};q^4)_\infty(q^3\zeta^2;q^4)_\infty}$$

and

$$\frac{\mathcal{N}_{\chi\Lambda_-}(\zeta_1,-iq^{\frac{1}{2}}\zeta_2)}{\mathcal{N}_{\Lambda_-\chi}(-iq^{\frac{1}{2}}\zeta_2,\zeta_1)} = -\zeta^{-1}\frac{(q^3\zeta^{-2};q^4)_\infty(q\zeta^2;q^4)_\infty}{(q\zeta^{-2};q^4)_\infty(q^3\zeta^2;q^4)_\infty}.$$

We then use this to obtain the following exchange relation[3].

$$\begin{aligned}
& \Phi^R_{NS}(\zeta_1)\psi^{NS}(\zeta_2) \\
= \;& \chi(\zeta_1)\left(\Lambda_+(iq^{-\frac{1}{2}}\zeta_2)+\Lambda_-(-iq^{\frac{1}{2}}\zeta_2)\right) && (3.73)\\
= \;& \frac{\mathcal{N}_{\chi\Lambda_+}(\zeta_1, iq^{-\frac{1}{2}}\zeta_2)}{\mathcal{N}_{\Lambda_+\chi}(iq^{-\frac{1}{2}}\zeta_2,\zeta_1)}\Lambda_+(iq^{-\frac{1}{2}}\zeta_2)\chi(\zeta_1) \\
& +\frac{\mathcal{N}_{\chi\Lambda_-}(\zeta_1, -iq^{\frac{1}{2}}\zeta_2)}{\mathcal{N}_{\Lambda_-\chi}(-iq^{\frac{1}{2}}\zeta_2,\zeta_1)}\Lambda_-(-iq^{\frac{1}{2}}\zeta_2)\chi(\zeta_1) \\
= \;& -\zeta^{-1}\frac{(q^3\zeta^{-2};q^4)_\infty(q\zeta^2;q^4)_\infty}{(q\zeta^{-2};q^4)_\infty(q^3\zeta^2;q^4)_\infty}\left(\Lambda_+(iq^{-\frac{1}{2}}\zeta_2)\chi(\zeta_1)+\Lambda_-(-iq^{\frac{1}{2}}\zeta_2)\chi(\zeta_1)\right) \\
= \;& -\zeta^{-1}\frac{(q^3\zeta^{-2};q^4)_\infty(q\zeta^2;q^4)_\infty}{(q\zeta^{-2};q^4)_\infty(q^3\zeta^2;q^4)_\infty}\psi^{NS}(\zeta_2)\Phi^{NS}_R(\zeta_1). && (3.74)
\end{aligned}$$

We obtain the same relation for $NS \leftrightarrow R$:

$$\Phi^{NS}_R(\zeta_1)\psi^R(\zeta_2) \;=\; -\zeta^{-1}\frac{(q^3\zeta^{-2};q^4)_\infty(q\zeta^2;q^4)_\infty}{(q\zeta^{-2};q^4)_\infty(q^3\zeta^2;q^4)_\infty}\psi^R(\zeta_2)\Phi^R_{NS}(\zeta_1). \quad (3.75)$$

These objects are related to our fermionic operators $\Omega^R_{NS}(z)$, $\Omega^{NS}_R(z)$ and $\phi^{NS/R}(w)$ through the identifications

$$\begin{aligned}
\Omega^R_{NS}(z) &= \Phi^R_{NS}(q^{-1}z^{-\frac{1}{2}}), && (3.76)\\
\Omega^{NS}_R(z) &= \Phi^{NS}_R(q^{-1}z^{-\frac{1}{2}}), && (3.77)\\
\phi^{R/NS}(w) &= \psi^{R/NS}(w^{-\frac{1}{2}}). && (3.78)
\end{aligned}$$

A straightforward substitution of these into (3.74) and (3.75) returns the exchange relation in its original form (3.27), as appears in [61] and [66], confirming that we have obtained a free field realisation of our fermion emission operators in terms of q-Virasoro oscillators.

[3]In [78], some typos appear, in particular in the equivalent exchange relations (95) and (96) on pg. 377.

3.7 Matrix Elements

Consider the matrix element of Ising vertex operators $\langle 0|\, \Phi_R^{NS}(\zeta_1)\Phi_{NS}^R(\zeta_2)\,|0\rangle$, where $|0\rangle \in H_0^0$. Then

$$\begin{aligned}
&\langle 0|\, \Phi_R^{NS}(\zeta_1)\Phi_{NS}^R(\zeta_2)\,|0\rangle \\
=\;& \beta \oint \frac{dz}{2\pi i}\frac{1}{z}\frac{\Theta_{-q^2}(iq^{\frac{1}{2}}\zeta_1/z)}{\Theta_{-q^2}(-iq^{\frac{1}{2}}\zeta_1/z)}\,\langle 0|\,\chi(\zeta_1)S_-(z)\chi(\zeta_2)\,|0\rangle \\
=\;& \beta \oint \frac{dz}{2\pi i}\frac{1}{z}\frac{\Theta_{-q^2}(iq^{\frac{1}{2}}\zeta_1/z)}{\Theta_{-q^2}(-iq^{\frac{1}{2}}\zeta_1/z)}\zeta_1^{\frac{1}{8}}\zeta_2^{\frac{1}{8}}z^{\frac{1}{2}}\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2)\mathcal{N}_{\chi S_-}(\zeta_1,z)\mathcal{N}_{S_-\chi}(z,\zeta_2) \\
=\;& \beta\frac{(-q\zeta^{-1};-q^2)_\infty(q^6\zeta^{-2};q^4;q^4)_\infty}{(q^4\zeta^{-2};q^4;q^4)_\infty}\zeta^{-\frac{1}{8}} \\
&\times \oint \frac{dz}{2\pi i}\frac{1}{z}\frac{(iq^{\frac{1}{2}}\zeta_1/z;-q^2)_\infty(iq^{\frac{3}{2}}z/\zeta_1;-q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2)_\infty(-iq^{\frac{1}{2}}z/\zeta_1;-q^2)_\infty}\frac{(-q^{\frac{3}{2}}i\zeta_2/z;-q^2)_\infty}{(-q^{\frac{1}{2}}i\zeta_2/z;-q^2)_\infty},
\end{aligned}$$

where

$$-q^{\frac{1}{2}}\zeta_i < z < -i\zeta_1 q^{-\frac{1}{2}}.$$

We will now focus on the computation of the integral

$$I = \oint \frac{dz}{2\pi i}\frac{1}{z}\frac{(iq^{\frac{1}{2}}\zeta_1/z;-q^2)_\infty(iq^{\frac{3}{2}}z/\zeta_1;-q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2)_\infty(-iq^{\frac{1}{2}}z/\zeta_1;-q^2)_\infty}\frac{(-q^{\frac{3}{2}}i\zeta_2/z;-q^2)_\infty}{(-q^{\frac{1}{2}}i\zeta_2/z;-q^2)_\infty}.$$

In the notation of [98], see Appendix E, we have an integral of the form

$$I_m = \oint_K \frac{dz}{2\pi i}z^{m-1}\frac{(a_1z,\ldots,a_Az,b_1/z,\ldots b_B/z;x)_\infty}{(c_1z,\ldots,c_Cz,d_1/z,\ldots d_D/z;x)_\infty},\quad |x|<1$$

with $m=0$, $x=-q^2$, $A=C=1$, $B=D=2$ and

$$\begin{aligned}
a_1 &= iq^{\frac{3}{2}}/\zeta_1,\quad b_1 = iq^{\frac{1}{2}}\zeta_1,\quad b_2 = -q^{\frac{3}{2}}i\zeta_2 \\
c_1 &= -iq^{\frac{1}{2}}\zeta_1,\quad d_1 = -iq^{\frac{1}{2}}\zeta_1,\quad d_2 = -q^{\frac{1}{2}}i\zeta_2.
\end{aligned}$$

The poles of $\frac{1}{(c_1 z;-q^2)_\infty}$ lie outside of K, whilst the poles of $\frac{1}{(d_1/z,d_2/z;-q^2)_\infty}$ lie inside the contour. We also see that

$$\left|\frac{a_1 q^{-m}}{c_1}\right| = |q| < 1,$$

and so we have an integral which can be computed using the Gasper-Rahman method for the second case in Appendix E. With this, we have

$$\begin{aligned} I &= \frac{(b_1c_1, b_2c_1, a_1/c_1; -q^2)_\infty}{(-q^2, d_1c_1, d_2c_1; -q^2)_\infty} {}_3\phi_2 \left[\begin{matrix} d_1c_1 & d_2c_1 & -q^2c_1/a_1 \\ & b_1c_1 & b_2c_1 \end{matrix} ; -q^2, a_1/c_1 \right] \\ &= \frac{(q, -q^2\zeta^{-1}, -q; -q^2)_\infty}{(-q^2, -q, -q\zeta^{-1}; -q^2)_\infty} {}_3\phi_2 \left[\begin{matrix} -q & -q\zeta^{-1} & q \\ q & -q^2\zeta^{-1} & \end{matrix} ; -q^2, -q \right] \\ &= \frac{(q, -q^2\zeta^{-1}; -q^2)_\infty}{(-q^2, -q\zeta^{-1}; -q^2)_\infty} {}_2\phi_1 \left[\begin{matrix} -q & -q\zeta^{-1} \\ & -q^2\zeta^{-1} \end{matrix} ; -q^2, -q \right], \end{aligned}$$

where the basic hypergeometric series is defined as

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_2; q, z) &\equiv {}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n. \end{aligned}$$

We then use Heine's transformation formula [98, 99]

$${}_2\phi_1(a, b; c; q, z) = \frac{(b, az, q)_\infty}{(c, z; q)_\infty} {}_2\phi_1(c/b, z; az; q, b),$$

to rewrite this as

$$\begin{aligned}
I &= \frac{(q,-q^2\zeta^{-1};-q^2)_\infty}{(-q^2,-q\zeta^{-1};-q^2)_\infty}\frac{(-q\zeta^{-1},q^2;-q^2)_\infty}{(-q^2\zeta^{-1},-q;-q^2)_\infty}\,{}_2\phi_1\left[\begin{array}{cc} q & -q \\ & q^2 \end{array};-q^2,-q\zeta^{-1}\right] \\
&= \frac{(q;-q^2)_\infty}{(-q^2;-q^2)_\infty}\frac{(q^2;-q^2)_\infty}{(-q;-q^2)_\infty}\sum_{n=1}^{\infty}\frac{(q;-q^2)_n(-q;-q^2)_n}{(-q^2;-q^2)_n(q^2;-q^2)_n}(-q\zeta^{-1})^n \\
&= \beta^{-1}\sum_{n=1}^{\infty}\frac{(q^2;q^4)_n}{(q^4;q^4)_n}(-q\zeta^{-1})^n \\
&= \beta^{-1}\frac{(-q^3\zeta^{-1};q^4)_\infty}{(-q\zeta^{-1};q^4)_\infty},
\end{aligned}$$

where in the last step, we have used the q-binomial theorem [99]

$${}_1\phi_0(a;q,z) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty},\ \ |z|<1,\ \ |q|<1.$$

For the two-point function, this gives

$$\begin{aligned}
\zeta^{\frac{1}{8}}\langle 0|\,\Phi_R^{NS}(\zeta_1)\Phi_{NS}^R(\zeta_2)\,|0\rangle &= \frac{(-q\zeta^{-1};-q^2)_\infty(q^6\zeta^{-2};q^4;q^4)_\infty}{(q^4\zeta^{-2};q^4;q^4)_\infty}\frac{(-q^3\zeta^{-1};q^4)_\infty}{(-q\zeta^{-1};q^4)_\infty} \\
&= \frac{(q^6\zeta^{-2};q^4,q^8)_\infty^2}{(q^4\zeta^{-2};q^4,q^4)_\infty}, \qquad (3.79)
\end{aligned}$$

which agrees with the result for the same two point function in [22] (with $\zeta \mapsto \zeta^{-1}$). The method for dealing with fermion emission operators used in [22] (and [61]) to obtain this result is the same as that used in [79] to obtain a Pfaffian expression for the fermionic contribution to the S^+ form-factor, which will be discussed in the following chapter. In [22], this explicit form for the two point function is conjectured, having been arrived at through expanding an unpleasant expression in terms of Pfaffians to high orders in q, in contrast with our *exact* result (3.79) obtained using Shiraishi's realisation for fermions.

3.8 Fermionic Trace Expressions for Form-Factors

In this section, we will apply the Shiraishi realisation to the fermionic part of the one boson, one fermion free field realisation. Within a $2m$-particle form-factor, our type II vertex operators will always appear in pairs $\Psi_{\ell_{2j-1}}(\xi_{2j-1})\Psi_{\ell_{2j}}(\xi_{2j})$, $1 \leq j \leq m$. One component of each pair will contain an Ω_{NS}^{R} fermion emission operator and the other will contain an Ω_{R}^{NS} fermion emission emission operator, the order in which they appear depending on whether we are taking a trace over a Ramond or a Neveu-Schwarz fermion sector. For the trace over $V(\lambda_0)$, $V(\lambda_2)$, we will be taking the trace over the Neveu-Schwarz fermion sector and so have pairs of fermion emission operators appearing with $\Omega_{NS}^{R}(w)$ to the left. If we instead choose $V(\lambda_1)$, the highest weight module has Ramond fermion sector and so we have $\Omega_{R}^{NS}(w)$ to the left in our pairs. There are four possible pairings of type II vertex operators we can encounter and we indicate the fermion contribution for each below.

$$
\begin{aligned}
\Psi_0(\xi_{2j-1})\Psi_0(\xi_{2j}) : &\quad \Omega(q^{-2}\xi_{2j-1})\Omega(q^{-2}\xi_{2j}) \\
\Psi_0(\xi_{2j-1})\Psi_1(\xi_{2j}) : &\quad \Omega(q^{-2}\xi_{2j-1})\Omega(q^{-2}\xi_{2j})\phi(w_{2j}) \\
\Psi_1(\xi_{2j-1})\Psi_0(\xi_{2j}) : &\quad \Omega(q^{-2}\xi_{2j-1})\phi(w_{2j-1})\Omega(q^{-2}\xi_{2j}) \\
\Psi_1(\xi_{2j-1})\Psi_1(\xi_{2j}) : &\quad \Omega(q^{-2}\xi_{2j-1})\phi(w_{2j-1})\Omega(q^{-2}\xi_{2j})\phi(w_{2j}).
\end{aligned}
$$

We will need to keep track of the order these pairings appear in the trace so that we can insert the correct normal ordering factors once we move to the Shiraishi realisation using (3.76), (3.77) and (3.78). We now introduce the following fermionic operators for $1 \leq j \leq m$:

$$
\begin{aligned}
\mathcal{O}_{\Psi_{00}}^{NS}(\xi_{2j-1},\xi_{2j}) &= \Phi_{R}^{NS}(\xi_{2j-1})\Phi_{NS}^{R}(\xi_{2j}), && (3.80)\\
\mathcal{O}_{\Psi_{01}}^{NS}(\xi_{2j-1},\xi_{2j},u_{2j}) &= \Phi_{R}^{NS}(\xi_{2j-1})\Phi_{NS}^{R}(\xi_{2j})\phi^{NS}(u_{2j}), && (3.81)\\
\mathcal{O}_{\Psi_{10}}^{NS}(\xi_{2j-1},u_{2j-1}\xi_{2j}) &= \Phi_{R}^{NS}(\xi_{2j-1})\psi^{R}(u_{2j-1})\Phi_{NS}^{R}(\xi_{2j}) && (3.82)\\
\mathcal{O}_{\Psi_{11}}^{NS}(\xi_{2j-1},u_{2j-1},\xi_{2j},u_{2j}) &= \Phi_{R}^{NS}(\xi_{2j-1})\psi^{R}(u_{2j-1})\Phi_{NS}^{R}(\xi_{2j})\phi^{NS}(u_{2j}), && (3.83)
\end{aligned}
$$

as well as the same operators with $NS \leftrightarrow R$.

For any combination of these operators, using the normal ordering relations and trace contributions detailed in Appendix D, it is possible to extract an integral expression for the trace over either H_0^0 or H_1^0, corresponding to either a Neveu Schwarz or Ramond trace. The integrands arising through this method are built from q-infinite products and so, if certain conditions hold, they are able to be computed analytically using the Gasper-Rahman [98] method detailed in Appendix E. If analytic computation is not possible, then as we are in the region $|q| < 1$, it will be possible to expand to high orders in q and extract the appropriate coefficient.

3.8.1 Two-particle Form-Factors

For the two-spinon contribution to the form-factors, we will only encounter a single pair of type II vertex operators. For simplicity, we consider form factors of the form

$$\begin{aligned} &\langle \mathrm{vac}| E^{\varepsilon}_{\varepsilon'} |\xi_1, \xi_2\rangle^{(i;\pm)}_{\ell_1,\ell_2} \\ \propto\ & \mathrm{Tr}_{V(\lambda_i)}\left(q^{-2\rho}\Phi_{2-\varepsilon}(q^{-2}z_1)\Phi_{\varepsilon'}(z_1)\Psi^*_{\ell_1}(\xi_1)\Psi^*_{\ell_2}(\xi_2)\right), \end{aligned} \tag{3.84}$$

so that we only have two type I vertex operators to consider. From this, we can build form-factors for local operators such as S^+, S^- and S^z. The bosonic and lattice contributions to the trace can easily be extracted from (3.65) and (3.48) in Sections 3.5 and 3.4.2 and so we now focus on the form of the trace of the fermionic part using Shiraishi's free field realisation for fermions. Along with the inclusion of factors from normalisation (3.26), (3.28), and shifts coming from the movement from dual vertex operators to normal vertex operators (3.30)-(3.33), we are considering a trace of the form

$$\mathrm{Tr}_{V(\lambda_i)}\left(q^{-2\rho}\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2)\Psi_{\ell_1}(\xi_1)\Psi_{\ell_2}(\xi_2)\right).$$

Restricting to only fermionic parts, we would like to compute

$$\mathrm{Tr}_{\mathcal{F}^{\phi^{NS}}}\left(\mathcal{O}^{NS}_{\Phi_{\varepsilon_1}}\mathcal{O}^{NS}_{\Phi_{\varepsilon_2}}\mathcal{O}^{NS}_{\Psi_{\ell_1\ell_2}}\right) \tag{3.85}$$

and

$$\mathrm{Tr}_{\mathcal{F}^{\phi^{R}}}\left(\mathcal{O}^{R}_{\Phi_{\varepsilon_1}}\mathcal{O}^{R}_{\Phi_{\varepsilon_2}}\mathcal{O}^{R}_{\Psi_{\ell_1\ell_2}}\right). \tag{3.86}$$

Writing down a general expression for these traces for arbitrary $\varepsilon_1, \varepsilon_2, \ell_1, \ell_2$ proves both difficult and tedious due to the different number of terms generated in each case from the additive nature of the $\psi(w)$ component (3.69), (3.70). In each specific case, however, all of the ingredients required to write down an explicit integral formula are given by (3.71)-(3.70) and Appendix D. In the next chapter, we will consider the specific case of the S^+ form-factor, where we are able to also compute the integral arising through the computation of the traces (3.85) and (3.86).

Chapter 4

Specialisation to the S^+ Form-Factor

In this chapter, we take the formalism set up in the previous chapter for the one boson, one fermion free field realisation and specialise the expressions in order to compute the 2-spinon contribution to the S^+ form-factor. By using Shiraishi's realisation for the fermion contribution to the trace, we are able to express the result as an explicit single contour integral. Using the trace expression (3.8) the two-particle form-factors (3.2) are proportional to the following trace

$$
{}^{(i)}\langle\text{vac}|E^{\varepsilon_2}_{\varepsilon_1}|\xi_1,\xi_2\rangle^{(i;\pm)}_{\ell_1,\ell_2} \quad \propto \quad F(z;\xi_1,\xi_2)^{(i,\pm)}_{2-\varepsilon_1,2-\varepsilon_2,\ell_1,\ell_2}, \tag{4.1}
$$

$$
F(z;\xi_1,\xi_2)^{(i,\pm)}_{\varepsilon_1,\varepsilon_2,\ell_1,\ell_2} \quad := \quad \mathrm{Tr}_{V(\lambda_i)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_i}_{\lambda_{2-i};\,\varepsilon_1}(zq^{-2})\tilde{\Phi}^{\lambda_{2-i}}_{\lambda_i;\,\varepsilon_2}(z)\tilde{\Psi}^{*\,\lambda_i}_{\lambda_{i\pm1};\,\ell_1}(\xi_1)\tilde{\Psi}^{*\,\lambda_{i\pm1}}_{\lambda_i;\,\ell_2}(\xi_2)\right).
$$

To be non-zero, form-factors $F(z;\xi_1,\xi_2)^{(i,\pm)}_{\varepsilon_1,\varepsilon_2,\ell_1,\ell_2}$ have the restriction $\varepsilon_1+\varepsilon_2+\ell_1+\ell_2=3$, and involve $N=4-\varepsilon_1-\varepsilon_2+\ell_1+\ell_2=1+2(\ell_1+\ell_2)$ integrals arising from the inclusion in the vertex operators of $U_q(\widehat{sl}_2)$ currents. The simplest case is when $N=1$, corresponding to $F(z;\xi_1,\xi_2)^{(i,\pm)}_{2,1,0,0}$ or $F(z;\xi_1,\xi_2)^{(i,\pm)}_{1,2,0,0}$. The sum of these two contributions is the S^+ form-factor:

$$
\langle\text{vac}|S^+|\xi_1,\xi_2\rangle^{(i;\pm)}_{\ell_1,\ell_2} \quad \propto \quad F(z;\xi_1,\xi_2)^{(i,\pm)}_{2,1,\ell_1,\ell_2}+F(z;\xi_1,\xi_2)^{(i,\pm)}_{1,2,\ell_1,\ell_2}, \tag{4.2}
$$

as we can write $S^+ = E_0^1 + E_1^2$. This is the object we now focus on computing explicitly, motivated by its relative simplicity and relation to the dynamical structure factors for spin-1.

4.1 Boson Contributions to the S^+ Form-Factor

In order to compute the bosonic part of the trace of both contributions to the form-factor (4.2), we follow the same process as in Section 3.4 and use (3.65). We are computing the traces

$$\mathrm{Tr}_{\mathcal{F}_a}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_i}_{\lambda_{2-i};\,1}(zq^{-2})\tilde{\Phi}^{\lambda_{2-i}}_{\lambda_i;\,2}(z)\tilde{\Psi}^{\lambda_i}_{\lambda_{i\pm1};\,0}(\xi_1)\tilde{\Psi}^{\lambda_{i\pm1}}_{\lambda_i;\,0}(\xi_2)\right), \tag{4.3}$$

and

$$\mathrm{Tr}_{\mathcal{F}_a}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_i}_{\lambda_{2-i};\,2}(zq^{-2})\tilde{\Phi}^{\lambda_{2-i}}_{\lambda_i;\,1}(z)\tilde{\Psi}^{\lambda_i}_{\lambda_{i\pm1};\,0}(\xi_1)\tilde{\Psi}^{\lambda_{i\pm1}}_{\lambda_i;\,0}(\xi_2)\right). \tag{4.4}$$

The i dependence of the trace of these objects over $\mathcal{F}_a$ is manifest only in the normalisation factors entering through (3.26) and (3.28), so for (4.3) we use (3.65) with $\mathcal{A} = \emptyset$, $\mathcal{B} = \{1\}$, $\mathcal{C} = \{2\}$ and $\mathcal{D} = \{1, 2, \}$, $\mathcal{E} = \emptyset$.

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^a}\left(q^{-4d^a}\,\hat{\Phi}_1(z_1)\Phi_2(z_2)\hat{\Psi}_0(\xi_1)\hat{\Psi}_0(\xi_2)\right)\\
=\;&\oint\frac{dw_1}{2\pi i}F_1(z_1,w_1)\mathcal{N}_{BD}(z_1,w_1;\xi_1)\mathcal{N}_{BD}(z_1,w_1;\xi_2)\mathcal{N}_{CD}(z_2;\xi_1)\mathcal{N}_{CD}(z_2;\xi_1)\\
&\times\frac{1}{(q^4;q^4)_\infty}\mathcal{N}_{BC}(z_1,w_1;z_2)\mathcal{N}_{DD}(\xi_1;\xi_2)\\
&\times g_1(z_1,z_2)g_1(z_2,z_1)g_1(z_1,z_1)g_1(z_2,z_2)g_1(w_1,w_1)\\
&\times g_2(\xi_1,\xi_2)g_2(\xi_2,\xi_1)g_2(\xi_1,\xi_1)g_2(\xi_2,\xi_2)\\
&\times g_{\Phi_2 x^-}(z_1,w_1)g_{\Phi_2 x^-}(z_2,w_1)g_{\Psi_0 x^-}(\xi_1,w_1)g_{\Psi_0 x^-}(\xi_2,w_1)\\
&\times g_{\Phi_2\Psi_0}(z_1,\xi_1)g_{\Phi_2\Psi_0}(z_1,\xi_2)g_{\Phi_2\Psi_0}(z_2,\xi_1)g_{\Phi_2\Psi_0}(z_2,\xi_2)
\end{aligned}$$

We now substitute in the appropriate factors, using Appendix C, and specialise to $z_2 = z$, $z_1 = q^{-2}z$. Some simplification occurs, using q-infinite product relations detailed in Appendix A, and we have the integral expression

$$
\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^a}\left(q^{-4d^a}\hat{\Phi}_1(q^{-2}z)\Phi_2(z)\hat{\Psi}_0(\xi_1)\hat{\Psi}_0(\xi_2)\right)\\
= \;& -\oint \frac{dw_1}{2\pi i}F_1(q^{-2}z,w_1)\frac{(-q^2\xi_1)^{1/4}(q^4;q^4)_\infty(q^6;q^4)_\infty^3}{q^4z(1-\xi_1/qz)(1-\xi_2/qz)(1-q^6z/w_1)}\\
&\times\frac{(q^4\xi_2/\xi_1;q^4,q^4)_\infty\,(\xi_2/\xi_1;q^4,q^4)_\infty}{(q^2\xi_2/\xi_1;q^4,q^4)_\infty^2}\frac{(q^8\xi_1/\xi_2;q^4,q^4,q^4)_\infty\,(q^4\xi_1/\xi_2;q^4,q^4,q^4)_\infty}{(q^6\xi_1/\xi_2;q^4,q^4,q^4)_\infty^2}\\
&\times\frac{(q^8\xi_2/\xi_1;q^4,q^4,q^4)_\infty\,(q^4\xi_2/\xi_1;q^4,q^4,q^4)_\infty}{(q^6\xi_2/\xi_1;q^4,q^4,q^4)_\infty^2}\frac{(q^8;q^4,q^4,q^4)_\infty\,(q^4;q^4,q^4,q^4)_\infty}{(q^6;q^4,q^4,q^4)_\infty^2}\\
&\times\frac{1}{(q^8z/w_1;q^4)_\infty(q^4w_1/z;q^4)_\infty}\frac{1}{(q^{10}z/w_1;q^4)_\infty(q^2w_1/z;q^4)_\infty}\\
&\times\frac{(q^3w_1/\xi_1;q^4,q^4)_\infty(q^3\xi_1/w_1;q^4,q^4)_\infty}{(q^5w_1/\xi_1;q^4,q^4)_\infty(q^5\xi_1/w_1;q^4,q^4)_\infty}\frac{(q^3w_1/\xi_2;q^4,q^4)_\infty(q^3\xi_2/w_1;q^4,x^2)_\infty}{(q^5w_1/\xi_2;q^4,q^4)_\infty(q^5\xi_2/w_1;q^4,q^4)_\infty}\\
&\times\frac{(q^7\xi_1/z;q^4,q^4)_\infty}{(q^5z/\xi_1;q^4,q^4)_\infty}\frac{(q^7\xi_2/z;q^4,q^4)_\infty}{(q^5z/\xi_2;q^4,q^4)_\infty}\frac{(q^9z/\xi_1;q^4,q^4)_\infty}{(q^3\xi_1/z;q^4,q^4)_\infty}\frac{(q^9z/\xi_2;q^4,q^4)_\infty}{(q^3z/\xi_2;q^4,q^4)_\infty}.
\end{aligned}
$$

For convenience in later sections, we define a term $\widehat{\mathcal{N}}_{12}(z,\xi_1,\xi_2)$ by recasting this as

$$
\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^a}\left(q^{-4d^a}\hat{\Phi}_1(q^{-2}z)\Phi_2(z)\hat{\Psi}_0(\xi_1)\hat{\Psi}_0(\xi_2)\right)\\
= \;& \oint\frac{dw}{2\pi i}\widehat{\mathcal{N}}_{12}(z,\xi_1,\xi_2,w).
\end{aligned}
\tag{4.5}
$$

Similarly, for (4.4), we can compute

$$
\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^a}\left(q^{-4d^a}\hat{\Phi}_2(q^{-2}z)\Phi_1(z)\hat{\Psi}_0(\xi_1)\hat{\Psi}_0(\xi_2)\right)\\
= \;& \oint\frac{dw}{2\pi i}\widehat{\mathcal{N}}_{21}(z,\xi_1,\xi_2,w),
\end{aligned}
\tag{4.6}
$$

where

$$\begin{aligned}
\widehat{\mathcal{N}}_{21} &= F_1(z,w_2)\mathcal{N}_{BD}(z,w_2;\xi_1)\mathcal{N}_{BD}(z,w_2;\xi_2)\mathcal{N}_{CD}(q^{-2}z;\xi_1)\mathcal{N}_{CD}(q^{-2}z;\xi_1) \\
&\frac{1}{(q^4;q^4)_\infty}\times\mathcal{N}_{CB}(q^{-2}z;z,w_2)\mathcal{N}_{DD}(\xi_1;\xi_2) \\
&\times g_1(q^{-2}z,z)g_1(z,q^{-2}z)g_1(q^{-2}z,q^{-2}z)g_1(z,z)g_1(w_2,w_2) \\
&\times g_2(\xi_1,\xi_2)g_2(\xi_2,\xi_1)g_2(\xi_1,\xi_1)g_2(\xi_2,\xi_2) \\
&\times g_{\Phi_2 x^-}(q^{-2}z,w_2)g_{\Phi_2 x^-}(z,w_2)g_{\Psi_0 x^-}(\xi_1,w_2)g_{\Psi_0 x^-}(\xi_2,w_2) \\
&\times g_{\Phi_2\Psi_0}(q^{-2}z,\xi_1)g_{\Phi_2\Psi_0}(q^{-2}z,\xi_2)g_{\Phi_2\Psi_0}(z,\xi_1)g_{\Phi_2\Psi_0}(z,\xi_2).
\end{aligned}$$

4.2 Fermion Contributions to the S^+ Form-Factor

We now focus on computing the fermionic traces that contribute to the S^+ form-factor. In the language of the Shiraishi realisation, using (3.76), (3.77) and (3.78), these are of the form

$$T1 = \mathrm{Tr}_{H_1^0}\left(x^{-\hat{\rho}}\psi^R(w)\Phi_{NS}^R(\zeta_1)\Phi_R^{NS}(\zeta_2)\right) \quad (4.7)$$

$$T2 = \mathrm{Tr}_{H_0^0}\left(x^{-\hat{\rho}}\psi^{NS}(w)\Phi_R^{NS}(\zeta_1)\Phi_{NS}^R(\zeta_2)\right), \quad (4.8)$$

for the different choices of ground state. Such a trace is computed in [66] for a particular ratio of ζ_1/ζ_2 in which things simplify, but this specialisation is not useful for our purposes and we must consider the general case.

As mentioned in the previous chapter, it is possible to compute such a trace using the alternative free field realisation for the fermion emission operators outlined in [61] and [22][1]. The result is complicated and a method for extracting a concrete, usable expression from it has thus far shown to be elusive. The complicated form of this result is what led to the consideration of alternative free field realisations for both the fermion emission operators (as in the Shiraishi realisation) and for

[1]This is primarily the work of R. A. Weston and details of the computation will appear in [79]

the bosonisation of $U_q(\widehat{sl}_2)$ itself (as will be seen in the following chapter on the q-Wakimoto scheme).

4.2.1 The Shiraishi Approach

We now focus on computing (4.7) and (4.8) using Shiraishi's free field realisation for fermions. Using (3.69)-(3.72), we can express the first trace as follows.

$$
\begin{aligned}
T_1 &= \mathrm{Tr}_{H_1^0}\left(x^{-\hat{\rho}}\psi^R(w)\Phi^R_{NS}(\zeta_1)\Phi^{NS}_R(\zeta_2)\right)\\
&= \alpha\mathrm{Tr}_{H_1^0}\left(x^{-\hat{\rho}}T(w)\Phi^R_{NS}(\zeta_1)\Phi^{NS}_R(\zeta_2)\right)\\
&= \alpha\beta\oint\frac{dz}{2\pi iz}\frac{\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_2/z\right)}{\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_2/z\right)}\\
&\quad\times\mathrm{Tr}_{H_1^0}\left(x^{-\hat{\rho}}\left(\Lambda_+(iq^{-\frac{1}{2}}w)+\Lambda_-(-iq^{\frac{1}{2}}w)\right)\chi(\zeta_1)\chi(\zeta_2)S_-(z)\right)\\
&= g_{\chi\chi}(\zeta_1,\zeta_2)\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2)g_\chi(\zeta_1)g_\chi(\zeta_2)g_{\Lambda_+}\frac{x^{\frac{1}{8}}}{(x;x)_\infty}\alpha\beta\\
&\quad\times\oint\frac{dz}{2\pi iz}g_{S_-\chi}(z,\zeta_1)\mathcal{N}_{\chi S_-}(\zeta_1,z)g_{S_-\chi}(z,\zeta_2)\mathcal{N}_{\chi S_-}(\zeta_2,\zeta)g_{S_-}(z)\\
&\quad\times\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_1/z\right)\Bigg(y_+\Theta_{x^4}(-xy_+^{-4})g_{\Lambda_+S_-}(iq^{-\frac{1}{2}}w,z)\mathcal{N}_{\Lambda_+S_-}(iq^{-\frac{1}{2}}w,z)\\
&\qquad\times g_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_1)\mathcal{N}_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_1)g_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_2)\mathcal{N}_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_2)\\
&\qquad-y_-\Theta_{x^4}(-xy_-^{-4})g_{\Lambda_-S_-}(-iq^{\frac{1}{2}}w,z)\mathcal{N}_{\Lambda_-S_-}(-iq^{\frac{1}{2}}w,z)g_{\Lambda_-\chi}(-iq^{\frac{1}{2}}w,\zeta_1)\\
&\qquad\times\mathcal{N}_{\Lambda_-\chi}(-iq^{\frac{1}{2}}w,\zeta_1)g_{\Lambda_-\chi}(-iq^{-\frac{1}{2}}w,\zeta_2)\mathcal{N}_{\Lambda_-\chi}(-iq^{-\frac{1}{2}}w,\zeta_2)\Bigg),
\end{aligned}
$$

where $y_\pm = q^{\pm\frac{1}{2}}(\zeta_1\zeta_2/z^2)^{\frac{1}{4}}$ and we note that $g_{\Lambda_+} = -g_{\Lambda_-}$. Focusing on the integral, we have

$$
\begin{aligned}
& \mathrm{Tr}_{H_1^0}\left(x^{-\hat{\rho}}\phi^R(w)\Phi^R_{NS}(\zeta_1)\Phi^{NS}_R(\zeta_2)\right)\left(g_{\chi\chi}(\zeta_1,\zeta_2)\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2)g_\chi(\zeta_1)g_\chi(\zeta_2)g_{\Lambda_+}\right)^{-1} \\
= \; & \frac{x^{\frac{1}{8}}}{(x;x)_\infty}\alpha\beta\oint\frac{dz}{2\pi i z}g_{S_-\chi}(z,\zeta_1)\mathcal{N}_{\chi S_-}(\zeta_1,z)g_{S_-\chi}(z,\zeta_2)\mathcal{N}_{\chi S_-}(\zeta_2,z)g_{S_-}(z) \\
& \times\frac{\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_2/z\right)}{\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_2/z\right)}\Bigg(y_+\Theta_{x^4}(-xy_+^{-4})g_{\Lambda_+S_-}(iq^{-\frac{1}{2}}w,z)\mathcal{N}_{\Lambda_+S_-}(iq^{-\frac{1}{2}}w,z) \\
& \quad\times g_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_1)\mathcal{N}_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_1)g_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_2)\mathcal{N}_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_2) \\
& \quad -y_-\Theta_{x^4}(-xy_-^{-4})g_{\Lambda_-S_-}(-iq^{\frac{1}{2}}w,z)\mathcal{N}_{\Lambda_-S_-}(-iq^{\frac{1}{2}}w,z)g_{\Lambda_-\chi}(-iq^{\frac{1}{2}}w,\zeta_1) \\
& \quad\times\mathcal{N}_{\Lambda_-\chi}(-iq^{\frac{1}{2}}w,\zeta_1)g_{\Lambda_-\chi}(-iq^{-\frac{1}{2}}w,\zeta_2)\mathcal{N}_{\Lambda_-\chi}(-iq^{-\frac{1}{2}}w,\zeta_2)\Bigg).
\end{aligned}
$$

Substituting in $x = q^2$ along with the appropriate normal ordering factors $\mathcal{N}_{\mathcal{O}_i\mathcal{O}_j}$ and trace contributions $g_{\mathcal{O}_i\mathcal{O}_j}$, we obtain

$$
\begin{aligned}
&\mathrm{Tr}_{H_1^0}\left(q^{-2\hat{\rho}}\phi^R(w)\Phi^R_{NS}(\zeta_1)\Phi^{NS}_R(\zeta_2)\right)\left(g_{\chi\chi}(\zeta_1,\zeta_2)\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2)g_\chi(\zeta_1)g_\chi(\zeta_2)g_{\Lambda_+}\right)^{-1} \\
= \;& (\zeta_1\zeta_2)^{\frac{1}{4}}\frac{q^{\frac{1}{4}}}{(q^2;q^2)_\infty}\alpha\beta\frac{(q^3;-q^2,q^2)_\infty(-q^4;-q^2,q^2)_\infty}{(q^2;-q^2,q^2)_\infty(-q^3;-q^2,q^2)_\infty} \\
&\times\oint\frac{dz}{2\pi i}\frac{(-iq^{\frac{7}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_1;-q^2,q^2)_\infty}{(-iq^{\frac{5}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{1}{2}}z/\zeta_1;-q^2,q^2)_\infty} \\
&\times\frac{(-iq^{\frac{7}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_2;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{1}{2}}z/\zeta_2;-q^2,q^2)_\infty}\frac{(iq^{\frac{1}{2}}\zeta_2/z;-q^2)_\infty(iq^{\frac{3}{2}}z/\zeta_2;-q^2)_\infty}{(-iq^{\frac{3}{2}}z/\zeta_2;-q^2)_\infty} \\
&\times\Bigg(q^{\frac{3}{2}}\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(-iq^3z/w;q^2,q^2)_\infty}{(-iq^2z/w;q^2,q^2)_\infty}\frac{(q^2iw/z;q^2,q^2)_\infty}{(iqw/z;q^2,q^2)_\infty}\left(\frac{1+iz/w}{1+iqz/w}\right) \\
&\times\frac{(-q^{\frac{7}{2}}\zeta_1/w;q^2,q^2)_\infty(q^{\frac{7}{2}}\zeta_1/w;q^2,q^2)_\infty}{(-q^{\frac{5}{2}}\zeta_1/w;q^2,q^2)_\infty(q^{\frac{9}{2}}\zeta_1/w;q^2,q^2)_\infty}\frac{(q^{\frac{5}{2}}w/\zeta_1;q^2,q^2)_\infty(-q^{\frac{5}{2}}w/\zeta_1;q^2,q^2)_\infty}{(q^{\frac{3}{2}}w/\zeta_1;q^2,q^2)_\infty(-q^{\frac{7}{2}}w/\zeta_1;q^2,q^2)_\infty} \\
&\times\frac{(-q^{\frac{7}{2}}\zeta_2/w;q^2,q^2)_\infty(q^{\frac{7}{2}}\zeta_2/w;q^2,q^2)_\infty}{(-q^{\frac{5}{2}}\zeta_2/w;q^2,x)_\infty(q^{\frac{9}{2}}\zeta_2/w;q^2,q^2)_\infty}\frac{(q^{\frac{5}{2}}w/\zeta_2;q^2,q^2)_\infty(-q^{\frac{5}{2}}w/\zeta_2;q^2,q^2)_\infty}{(q^{\frac{3}{2}}w/\zeta_2;q^2,q^2)_\infty(-q^{\frac{7}{2}}w/\zeta_2;q^2,q^2)_\infty} \\
&\times q^{-1}\frac{(q^3\zeta_2^2/w^2;q^4)_\infty}{(1+q^{\frac{1}{2}}\zeta_2/w)(q^5\zeta_2^2/w^2;q^4)_\infty}\frac{(q^3\zeta_1^2/w^2;q^4)_\infty}{(1+q^{\frac{1}{2}}\zeta_1/w)(q^5\zeta_1^2/w^2;q^4)_\infty} \\
&-q^{-\frac{3}{2}}\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(iqz/w;q^2,q^2)_\infty}{(iq^2z/w);q^2,q^2)_\infty}\frac{(-iq^2w/z;q^2,q^2)_\infty}{(-iq^3w/z;q^2,q^2)_\infty}\left(\frac{1-iz/w}{1-iq^{-1}z/w}\right) \\
&\times\frac{(q^{\frac{3}{2}}\zeta_1/w;q^2,q^2)_\infty(-q^{\frac{7}{2}}\zeta_1/w;q^2,q^2)_\infty}{(q^{\frac{5}{2}}\zeta_1/w;q^2,q^2)_\infty(-q^{\frac{5}{2}}\zeta_1/w;q^2,q^2)_\infty}\frac{(-q^{\frac{5}{2}}w/\zeta_1;q^2,q^2)_\infty(q^{\frac{9}{2}}w/\zeta_1;q^2,q^2)_\infty}{(-q^{\frac{7}{2}}w/\zeta_1;q^2,q^2)_\infty(q^{\frac{7}{2}}w/\zeta_1;q^2,q^2)_\infty} \\
&\times\frac{(q^{\frac{3}{2}}\zeta_2/w;q^2,q^2)_\infty(-q^{\frac{7}{2}}\zeta_2/w;q^2,q^2)_\infty}{(q^{\frac{5}{2}}\zeta_2/w;q^2,q^2)_\infty(-q^{\frac{5}{2}}\zeta_2/w;q^2,q^2)_\infty}\frac{(-q^{\frac{5}{2}}w/\zeta_2;q^2,q^2)_\infty(q^{\frac{9}{2}}w/\zeta_2;q^2,q^2)_\infty}{(-q^{\frac{7}{2}}w/\zeta_2;q^2,q^2)_\infty(q^{\frac{7}{2}}w/\zeta_2;q^2,q^2)_\infty} \\
&\times q\frac{(1-q^{-\frac{1}{2}}\zeta_1/w)(q^3\zeta_1^2/w^2;q^4)_\infty}{(q\zeta_1^2/w^2;q^4)_\infty}\frac{(1-q^{-\frac{1}{2}}\zeta_2/w)(q^3\zeta_2^2/w^2;q^4)_\infty}{(q\zeta_2^2/w^2;q^4)_\infty}\Bigg).
\end{aligned}
$$

Ignoring pre factors, the two integrals we must attempt to compute are

$$
\begin{aligned}
I^R_{\Lambda_+}(\zeta_1,\zeta_2,w) \;=\;& q^{\frac{3}{2}}\oint_{K_1}\frac{dz}{2\pi i}\frac{(-iq^{\frac{7}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_1;-q^2,q^2)_\infty}{(-iq^{\frac{5}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{1}{2}}z/\zeta_1;-q^2,q^2)_\infty} \\
&\times\frac{(-iq^{\frac{7}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_2;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{1}{2}}z/\zeta_2;-q^2,q^2)_\infty} \\
&\times\frac{(iq^{\frac{1}{2}}\zeta_2/z;-q^2)_\infty(iq^{\frac{3}{2}}z/\zeta_2;-q^2)_\infty}{(-iq^{\frac{3}{2}}z/\zeta_2;-q^2)_\infty}\left(\frac{1+iz/w}{1+iqz/w}\right) \\
&\times\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(-iq^3z/w;q^2,q^2)_\infty}{(-iq^2z/w;q^2,q^2)_\infty}\frac{(q^2iw/z;q^2,q^2)_\infty}{(iqw/z;q^2,q^2)_\infty}
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
I^R_{\Lambda_-}(\zeta_1,\zeta_2,w) \;=\;& q^{-\frac{3}{2}}\oint_{K_2}\frac{dz}{2\pi i}\frac{(-iq^{\frac{7}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_1;-q^2,q^2)_\infty}{(-iq^{\frac{5}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{1}{2}}z/\zeta_1;-q^2,q^2)_\infty}\\
&\times\frac{(-iq^{\frac{7}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_2;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{1}{2}}z/\zeta_2;-q^2,q^2)_\infty}\\
&\times\frac{(iq^{\frac{1}{2}}\zeta_2/z;-q^2)_\infty(iq^{\frac{3}{2}}z/\zeta_2;-q^2)_\infty}{(-iq^{\frac{3}{2}}z/\zeta_2;-q^2)_\infty}\left(\frac{1-iz/w}{1-iq^{-1}z/w}\right)\\
&\times\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(iqz/w;q^2,q^2)_\infty}{(iq^2z/w;q^2,q^2)_\infty}\frac{(-iq^2w/z;q^2,q^2)_\infty}{(-iq^3w/z;q^2,q^2)_\infty},
\end{aligned}
\tag{4.10}
$$

where K_1 is the contour with poles of

$$\frac{1}{(-iq^{\frac{5}{2}}\zeta_1/z;-q^2,q^2)_\infty},\quad \frac{1}{(-iq^{\frac{1}{2}}\zeta_2/z;-q^2,q^2)_\infty}\quad\text{and}\quad\frac{1}{(iqw/z;q^2,q^2)_\infty}$$

lying inside (and all other poles lying outside). Similarly, K_2 is the contour with poles of

$$\frac{1}{(-iq^{\frac{5}{2}}\zeta_1/z;-q^2,q^2)_\infty},\quad \frac{1}{(-iq^{\frac{1}{2}}\zeta_2/z;-q^2,q^2)_\infty}\quad\text{and}\quad\frac{1}{(-iq^3w/z;q^2,q^2)_\infty}$$

lying inside (and all other poles lying outside). To calculate (4.9) and (4.10) analytically, we need to compute the residue

$$\operatorname*{Res}_{z=ab^nc^m}\left(\frac{1}{(a/z;b,c)_\infty}\right).$$

For $n \geq 1$, $m \geq 1$, we have

$$
\begin{aligned}
\operatorname*{Res}_{z=ab^nc^m}\left(\frac{1}{(a/z;b,c)_\infty}\right) &= \operatorname*{Res}_{z=ab^nc^m}\left(\frac{1}{\prod\limits_{j,k=0}^{\infty}(1-ab^jc^k/z)}\right) \\
&= \frac{ab^nc^m}{(1/b^nc^m;b)_n(1/c^m;b)_\infty}\frac{1}{(1/b^nc^{m-1};b)_n(1/c^{m-1};b)_\infty} \\
&\quad \ldots \\
&\quad \times\frac{1}{(1/b^nc;b)_n(1/c;b)_\infty}\frac{1}{(1/b^n;b)_n(b;b)_\infty}\frac{1}{(c/b^n;b)_n(c;b)_\infty} \\
&\quad \ldots \\
&\quad \times\frac{1}{(c^m/b^n;b)_n(c^m;b)_\infty}\frac{1}{(c^{m+1}/b^n;b)_n(c^{m+1};b)_\infty} \\
&\quad \ldots
\end{aligned}
$$

$$
\begin{aligned}
\operatorname*{Res}_{z=ab^nc^m}\left(\frac{1}{(a/z;b,c)_\infty}\right) &= \operatorname*{Res}_{z=ab^nc^m}\left(\frac{1}{\prod\limits_{j,k=0}^{\infty}(1-ab^jc^k/z)}\right) \\
&= \frac{ab^nc^m}{(1/b^nc^m;b,c)_{n,m}(1/c^m;c,b)_{m,\infty}}\frac{1}{(1/b^n;b,c)_{n,\infty}(b;b)_\infty}\frac{1}{(c;b,c)_\infty} \\
&= \frac{ab^nc^m}{(1/b^nc^m;b,c)_{n,m}(1/c^m;c,b)_{m,\infty}(1/b^n;b,c)_{n,\infty}} \\
&\quad \times\frac{1}{(b;b)_\infty(c;c)_\infty(bc;b,c)_\infty},
\end{aligned}
\tag{4.11}
$$

where the notation

$$\prod_{j=0}^{n}\prod_{k=0}^{m}(1-ab^jc^k) = (a;b,c)_{n,m} \quad ,$$

has been introduced. For cases when one or both of n and m are zero, we have

$$
\begin{aligned}
\operatorname*{Res}_{z=a}\left(\frac{1}{(a/z;b,c)_\infty}\right) &= \frac{a}{(b;b)_\infty(c;c)_\infty(bc;b,c)_\infty}, \\
\operatorname*{Res}_{z=ab^n}\left(\frac{1}{(a/z;b,c)_\infty}\right) &= \frac{ab}{(1/b^n;b,c)_{n,\infty}(b;b)_\infty(c;c)_\infty(bc;b,c)_\infty}, \\
\operatorname*{Res}_{z=ac^m}\left(\frac{1}{(a/z;b,c)_\infty}\right) &= \frac{ac}{(1/c^m;c,b)_{m,\infty}(b;b)_\infty(c;c)_\infty(bc;b,c)_\infty}.
\end{aligned}
$$

Using this, it is possible to explicitly compute both integrals (4.9) and (4.10) as double residue sums. The results are rather long and messy looking (they are given in full in Appendix F), but they are exact results in terms of sums of q-infinite products. It would be nice to find some generalisation of q-hypergeometric series that is applicable to the double sum appearing, but we have not yet been able to locate anything in the existing literature. To be as concise as possible, we now define

$$\begin{aligned}
& G_{\Lambda_+}(\zeta_1,\zeta_2,w) \\
= \; & q^{-1} g_{\chi\chi}(\zeta_1,\zeta_2)\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2) g_\chi(\zeta_1) g_\chi(\zeta_2) g_{\Lambda_+} \frac{(q^3;-q^2,q^2)_\infty(-q^4;-q^2,q^2)_\infty}{(q^2;-q^2,q^2)_\infty(-q^3;-q^2,q^2)_\infty} \\
& \times \frac{(q^7\zeta_1^2/w^2;q^4,q^4)_\infty}{(-q^{\frac{5}{2}}\zeta_1/w;q^2)_\infty(q^9\zeta_1^2/w^2;q^4,q^4)_\infty} \frac{(q^5w^2/\zeta_1^2;q^4,q^4)_\infty}{(q^{\frac{3}{2}}w/\zeta_1;q^2)_\infty(q^7w^2/\zeta_1^2;q^4,q^4)_\infty} \\
& \times \frac{(q^7\zeta_2^2/w^2;q^4,q^4)_\infty}{(-q^{\frac{5}{2}}\zeta_2/w;q^2)_\infty(q^9\zeta_2^2/w^2;q^4,q^4)_\infty} \frac{(q^5w^2/\zeta_2^2;q^4,q^4)_\infty}{(q^{\frac{3}{2}}w/\zeta_2;q^2)_\infty(q^7w^2/\zeta_2^2;q^4,q^4)_\infty} \\
& \times \frac{(q^3\zeta_2^2/w^2;q^4)_\infty}{(1+q^{\frac{1}{2}}\zeta_2/w)(q^5\zeta_2^2/w^2;q^4)_\infty} \frac{(q^3\zeta_1^2/w^2;q^4)_\infty}{(1+q^{\frac{1}{2}}\zeta_1/w)(q^5\zeta_1^2/w^2;q^4)_\infty}
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
& G_{\Lambda_-}(\zeta_1,\zeta_2,w) \\
= \; & q g_{\chi\chi}(\zeta_1,\zeta_2)\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2) g_\chi(\zeta_1) g_\chi(\zeta_2) g_{\Lambda_+} \frac{(q^3;-q^2,q^2)_\infty(-q^4;-q^2,q^2)_\infty}{(q^2;-q^2,q^2)_\infty(-q^3;-q^2,q^2)_\infty} \\
& \times \frac{(q^{\frac{3}{2}}\zeta_1/w;q^2)_\infty(q^7\zeta_1^2/w^2;q^4,q^4)_\infty}{(q^5\zeta_1^2/w^2;q^4,q^4)_\infty} \frac{(-q^{\frac{5}{2}}w/\zeta_1;q^2)_\infty(q^9w^2/\zeta_1^2;q^4,q^4)_\infty}{(q^7w^2/\zeta_1^2;q^4,q^4)_\infty} \\
& \times \frac{(q^{\frac{3}{2}}\zeta_2/w;q^2)_\infty(q^7\zeta_2^2/w^2;q^4,q^4)_\infty}{(q^5\zeta_2^2/w^2;q^4,q^4)_\infty} \frac{(-q^{\frac{5}{2}}w/\zeta_2;q^2)_\infty(q^9w^2/\zeta_2^2;q^4,q^4)_\infty}{(q^7w^2/\zeta_2^2;q^4,q^4)_\infty} \\
& \times \frac{(1-\zeta_1/q^{\frac{1}{2}}w)(q^3\zeta_1^2/w^2;q^4)_\infty}{(q\zeta_1^2/w^2;q^4)_\infty} \frac{(1-\zeta_2/q^{\frac{1}{2}}w)(q^3\zeta_2^2/w^2;q^4)_\infty}{(q\zeta_2^2/w^2;q^4)_\infty}.
\end{aligned} \tag{4.13}$$

With this, we can write the trace (4.7) for $x = q^2$ as

$$\begin{aligned}
& \mathrm{Tr}_{H_1^0}\left(q^{-2\hat{\rho}}\psi^R(w)\Phi_{NS}^R(\zeta_1)\Phi_R^{NS}(\zeta_2)\right) \\
= \; & \frac{(\zeta_1\zeta_2 q)^{\frac{1}{4}}}{(q^2;q^2)_\infty}\alpha\beta\left(G_{\Lambda_+}(\zeta_1,\zeta_2,w)I_{\Lambda_+}^R(\zeta_1,\zeta_2,w) - G_{\Lambda_-}(\zeta_1,\zeta_2,w)I_{\Lambda_-}^R(\zeta_1,\zeta_2,w)\right).
\end{aligned}$$

Using the identifications (3.76), (3.76) and (3.78), we are now able to write down the fermionic contribution to the S^+ form-factor for the choice of ground state

$V(\lambda_1)$.

$$\begin{aligned}
& \mathrm{Tr}_{\mathcal{F}_{\phi^R}}\left(q^{-2d_{\phi^R}}\phi(w)\Omega(q^{-2}\xi_1)\Omega(q^{-2}\xi_2)\right)\\
= & \mathrm{Tr}_{H_1^0}\left(x^{-\hat{\rho}}\psi^R(w^{-\frac{1}{2}})\Phi_{NS}^R(\xi_1^{-\frac{1}{2}})\Phi_R^{NS}(\xi_1^{-\frac{1}{2}})\right)\\
= & (\zeta_1\zeta_2)^{\frac{1}{4}}\frac{q^{\frac{1}{4}}}{(q^2;q^2)_\infty}\alpha\beta\Bigg(G_{\Lambda_+}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I_{\Lambda_+}^R(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\\
& -G_{\Lambda_-}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I_{\Lambda_-}^R(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\Bigg).
\end{aligned} \tag{4.14}$$

The computation of (4.8) works in exactly the same way and we are again left with a sum of two integrals of infinite products which we are able to compute by summing over the appropriate residues. Explicitly, we have

$$\begin{aligned}
T_2 = & \mathrm{Tr}_{H_0^0}\left(x^{-\hat{\rho}}\psi^R(w)\Phi_R^{NS}(\zeta_1)\Phi_{NS}^R(\zeta_2)\right)\\
= & \alpha\mathrm{Tr}_{H_0^0}\left(x^{-\hat{\rho}}T(w)\Phi_R^{NS}(\zeta_1)\Phi_{NS}^R(\zeta_2)\right)\\
= & \alpha\beta\oint\frac{dz}{2\pi iz}\frac{\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_1/z\right)}{\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_1/z\right)}\\
& \times\mathrm{Tr}_{H_1^0}\left(x^{-\hat{\rho}}\left(\Lambda_+(iq^{-\frac{1}{2}}w)+\Lambda_-(-iq^{\frac{1}{2}}w)\right)\chi(\zeta_1)S_-(z)\chi(\zeta_2)\right)\\
= & g_{\chi\chi}(\zeta_1,\zeta_2)\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2)g_\chi(\zeta_1)g_\chi(\zeta_2)g_{\Lambda_+}\frac{x^{\frac{1}{8}}}{(x;x)_\infty}\alpha\beta\\
& \times\oint\frac{dz}{2\pi iz}g_{S_-\chi}(z,\zeta_1)\mathcal{N}_{\chi S_-}(\zeta_1,z)g_{S_-\chi}(z,\zeta_2)\mathcal{N}_{S_-\chi}(z,\zeta_2)g_{S_-}(z)\\
& \times\frac{\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_1/z\right)}{\Theta_{-q^2}\left(iq^{\frac{1}{2}}\xi_1/z\right)}\Bigg(\Theta_{x^4}(-x^2y_+^{-4})g_{\Lambda_+S_-}(iq^{-\frac{1}{2}}w,z)\mathcal{N}_{\Lambda_+S_-}(iq^{-\frac{1}{2}}w,z)\\
& \times g_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_1)\mathcal{N}_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_1)g_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_2)\mathcal{N}_{\Lambda_+\chi}(iq^{-\frac{1}{2}}w,\zeta_2)\\
& -\Theta_{x^4}(-x^2y_-^{-4})g_{\Lambda_-S_-}(-iq^{\frac{1}{2}}w,z)\mathcal{N}_{\Lambda_-S_-}(-iq^{\frac{1}{2}}w,z)g_{\Lambda_-\chi}(-iq^{\frac{1}{2}}w,\zeta_1)\\
& \times\mathcal{N}_{\Lambda_-\chi}(-iq^{\frac{1}{2}}w,\zeta_1)g_{\Lambda_-\chi}(-iq^{-\frac{1}{2}}w,\zeta_2)\mathcal{N}_{\Lambda_-\chi}(-iq^{-\frac{1}{2}}w,\zeta_2)\Bigg).
\end{aligned}$$

For $x = q^2$, this becomes the following.

$$
\begin{aligned}
& \mathrm{Tr}_{H_0^0}\left(q^{-2\hat{\rho}}\psi^R(w)\Phi_R^{NS}(\zeta_1)\Phi_{NS}^R(\zeta_2)\right)\left(g_{\chi\chi}(\zeta_1,\zeta_2)\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2)g_\chi(\zeta_1)g_\chi(\zeta_2)g_{\Lambda_+}\right)^{-1} \\
= \quad & \frac{q^{\frac{1}{4}}}{(q^2;q^2)_\infty}\alpha\beta\frac{(q^3;-q^2,q^2)_\infty(-q^4;-q^2,q^2)_\infty}{(q^2;-q^2,q^2)_\infty(-q^3;-q^2,q^2)_\infty} \\
& \times\oint\frac{dz}{2\pi iz}\frac{(-iq^{\frac{3}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{7}{2}}z/\zeta_2;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{5}{2}}z/\zeta_2;-q^2,q^2)_\infty} \\
& \times\frac{(-iq^{\frac{7}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_1;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{5}{2}}z/\zeta_1;-q^2,q^2)_\infty}\frac{(iq^{\frac{1}{2}}\zeta_1/z;-q^2)_\infty(iq^{\frac{3}{2}}z/\zeta_1;-q^2)_\infty}{(-iq^{\frac{32}{z}}/\zeta_1;-q^2)_\infty} \\
& \times\Bigg(\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(-iq^3z/w;q^2,q^2)_\infty}{(-iq^2z/w;q^2,q^2)_\infty}\frac{(q^2iw/z;q^2,q^2)_\infty}{(iqw/z;q^2,q^2)_\infty}\left(\frac{1+iz/w}{1+iqz/w}\right) \\
& \times\frac{(-q^{\frac{7}{2}}\zeta_1/w;q^2,q^2)_\infty(q^{\frac{7}{2}}\zeta_1/w;q^2,q^2)_\infty}{(-q^{\frac{5}{2}}\zeta_1/w;q^2,q^2)_\infty(q^{\frac{9}{2}}\zeta_1/w;q^2,q^2)_\infty}\frac{(q^{\frac{5}{2}}w/\zeta_1;q^2,q^2)_\infty(-q^{\frac{5}{2}}w/\zeta_1;q^2,q^2)_\infty}{(q^{\frac{3}{2}}w/\zeta_1;q^2,q^2)_\infty(-q^{\frac{7}{2}}w/\zeta_1;q^2,q^2)_\infty} \\
& \times\frac{(-q^{\frac{7}{2}}\zeta_2/w;q^2,q^2)_\infty(q^{\frac{7}{2}}\zeta_2/w;q^2,q^2)_\infty}{(-q^{\frac{5}{2}}\zeta_2/w;q^2,x)_\infty(q^{\frac{9}{2}}\zeta_2/w;q^2,q^2)_\infty}\frac{(q^{\frac{5}{2}}w/\zeta_2;q^2,q^2)_\infty(-q^{\frac{5}{2}}w/\zeta_2;q^2,q^2)_\infty}{(q^{\frac{3}{2}}w/\zeta_2;q^2,q^2)_\infty(-q^{\frac{7}{2}}w/\zeta_2;q^2,q^2)_\infty} \\
& \times q^{-1}\frac{(q^3\zeta_2^2/w^2;q^4)_\infty}{(1+q^{\frac{1}{2}}\zeta_2/w)(q^5\zeta_2^2/w^2;q^4)_\infty}\frac{(q^3\zeta_1^2/w^2;q^4)_\infty}{(1+q^{\frac{1}{2}}\zeta_1/w)(q^5\zeta_1^2/w^2;q^4)_\infty} \\
& \quad -\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(iqz/w;q^2,q^2)_\infty}{(iq^2z/w);q^2,q^2)_\infty}\frac{(-iq^2w/z;q^2,q^2)_\infty}{(-iq^3w/z;q^2,q^2)_\infty}\left(\frac{1-iz/w}{1-iq^{-1}z/w}\right) \\
& \times\frac{(q^{\frac{3}{2}}\zeta_1/w;q^2,q^2)_\infty(-q^{\frac{7}{2}}\zeta_1/w;q^2,q^2)_\infty}{(q^{\frac{5}{2}}\zeta_1/w;q^2,q^2)_\infty(-q^{\frac{5}{2}}\zeta_1/w;q^2,q^2)_\infty}\frac{(-q^{\frac{5}{2}}w/\zeta_1;q^2,q^2)_\infty(q^{\frac{9}{2}}w/\zeta_1;q^2,q^2)_\infty}{(-q^{\frac{7}{2}}w/\zeta_1;q^2,q^2)_\infty(q^{\frac{7}{2}}w/\zeta_1;q^2,q^2)_\infty} \\
& \times\frac{(q^{\frac{3}{2}}\zeta_2/w;q^2,q^2)_\infty(-q^{\frac{7}{2}}\zeta_2/w;q^2,q^2)_\infty}{(q^{\frac{5}{2}}\zeta_2/w;q^2,q^2)_\infty(-q^{\frac{5}{2}}\zeta_2/w;q^2,q^2)_\infty}\frac{(-q^{\frac{5}{2}}w/\zeta_2;q^2,q^2)_\infty(q^{\frac{9}{2}}w/\zeta_2;q^2,q^2)_\infty}{(-q^{\frac{7}{2}}w/\zeta_2;q^2,q^2)_\infty(q^{\frac{7}{2}}w/\zeta_2;q^2,q^2)_\infty} \\
& \times q\frac{(1-q^{-\frac{1}{2}}\zeta_1/w)(q^3\zeta_1^2/w^2;q^4)_\infty}{(q\zeta_1^2/w^2;q^4)_\infty}\frac{(1-q^{-\frac{1}{2}}\zeta_2/w)(q^3\zeta_2^2/w^2;q^4)_\infty}{(q\zeta_2^2/w^2;q^4)_\infty}\Bigg).
\end{aligned}
$$

The integrals to consider in this case are

$$
\begin{aligned}
I_{\Lambda_+}^{NS}(\zeta_1,\zeta_2,w) \quad = \quad & \times\oint_{\hat{K}_1}\frac{dz}{2\pi iz}\frac{(-iq^{\frac{3}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{7}{2}}z/\zeta_2;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{5}{2}}z/\zeta_2;-q^2,q^2)_\infty} \\
& \times\frac{(-iq^{\frac{7}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_1;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{5}{2}}z/\zeta_1;-q^2,q^2)_\infty} \\
& \times\frac{(iq^{\frac{1}{2}}\zeta_1/z;-q^2)_\infty(iq^{\frac{3}{2}}z/\zeta_1;-q^2)_\infty}{(-iq^{\frac{3}{2}}z/\zeta_1;-q^2)_\infty}\left(\frac{1+iz/w}{1+iqz/w}\right) \\
& \times\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(-iq^3z/w;q^2,q^2)_\infty}{(-iq^2z/w;q^2,q^2)_\infty}\frac{(q^2iw/z;q^2,q^2)_\infty}{(iqw/z;q^2,q^2)_\infty}
\end{aligned}
\tag{4.15}
$$

and

$$\begin{aligned} I^{NS}_{\Lambda_-}(\zeta_1,\zeta_2,w) \;=\; & \times\oint_{\hat{K}_2}\frac{dz}{2\pi iz}\frac{(-iq^{\frac{3}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{7}{2}}z/\zeta_2;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_2/z;-q^2,q^2)_\infty(-iq^{\frac{5}{2}}z/\zeta_2;-q^2,q^2)_\infty} \\ & \times\frac{(-iq^{\frac{7}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{3}{2}}z/\zeta_1;-q^2,q^2)_\infty}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2,q^2)_\infty(-iq^{\frac{5}{2}}z/\zeta_1;-q^2,q^2)_\infty} \\ & \times\frac{(iq^{\frac{1}{2}}\zeta_1/z;-q^2)_\infty(iq^{\frac{3}{2}}z/\zeta_1;-q^2)_\infty}{(-iq^{\frac{3}{2}}z/\zeta_1;-q^2)_\infty}\left(\frac{1-iz/w}{1-iq^{-1}z/w}\right) \\ & \times\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(iqz/w;q^2,q^2)_\infty}{(iq^2z/w);q^2,q^2)_\infty}\frac{(-iq^2w/z;q^2,q^2)_\infty}{(-iq^3w/z;q^2,q^2)_\infty}, \end{aligned} \tag{4.16}$$

where the contour $\hat{K}_1$ has poles of

$$\frac{1}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2,q^2)_\infty},\quad \frac{1}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2,q^2)_\infty}\quad\text{and}\quad\frac{1}{(iqw/z;q^2,q^2)_\infty}$$

lying inside (with all other poles lying outside) and $\hat{K}_2$ is the contour with poles of

$$\frac{1}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2,q^2)_\infty},\quad \frac{1}{(-iq^{\frac{1}{2}}\zeta_1/z;-q^2,q^2)_\infty}\quad\text{and}\quad\frac{1}{(-iq^3w/z;q^2,q^2)_\infty}$$

lying inside (with all other poles lying outside). We can use the same method as in the case of (4.9) and (4.10) in order to compute the residue sum and evaluate the integrals explicitly. The trace (4.8) is then given by

$$\begin{aligned} & \mathrm{Tr}_{H_0^0}\left(q^{-2\hat{\rho}}\psi^{NS}(w)\Phi^{NS}_R(\zeta_1)\Phi^R_{NS}(\zeta_2)\right) \\ = \; & \frac{q^{\frac{1}{4}}}{(q^2;q^2)_\infty}\alpha\beta\left(G_{\Lambda_+}(\zeta_1,\zeta_2,w)I^{NS}_{\Lambda_+}(\zeta_1,\zeta_2,w)-G_{\Lambda_-}(\zeta_1,\zeta_2,w)I^{NS}_{\Lambda_-}(\zeta_1,\zeta_2,w)\right). \end{aligned}$$

Using identifications (3.76), (3.77) and (3.78), the fermionic contribution to the S^+ form-factor for the ground states $V(\lambda_0)$ and $V(\lambda_2)$ takes the form

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^{\phi NS}}\left(q^{-2d^{\phi^{NS}}}\phi^{NS}(w)\Omega_R^{NS}(q^{-2}\xi_1)\Omega_{NS}^R(q^{-2}\xi_2)\right)\\
&=\mathrm{Tr}_{H_0^0}\left(q^{-2\hat{\rho}}\psi^{NS}(w^{-\frac{1}{2}})\Phi_R^{NS}(\xi_1^{-\frac{1}{2}})\Phi_{NS}^R(\xi_2^{-\frac{1}{2}})\right)\\
&=\frac{q^{\frac{1}{4}}}{(q^2;q^2)_\infty}\alpha\beta\Bigg(G_{\Lambda_+}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I_{\Lambda_+}^{NS}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\\
&-G_{\Lambda_-}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I_{\Lambda_-}^{NS}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\Bigg).
\end{aligned}\tag{4.17}$$

4.3 The Final Integral Result

4.3.1 The $V(\lambda_1)$ Result

For the choice of ground state $V(\lambda_1)$, we have free field realisation (3.17). The fermionic trace is taken over the Ramond sector and so the fermion contribution is from (4.7). For the first term of the S^+ form-factor the bosonic contribution is given in (4.5) and the lattice contribution, recalling Section 3.4.2, is from

$$w^{\frac{1}{2}}\mathrm{Tr}_{e^{\frac{\alpha}{2}}\mathbb{C}[Q]}\left(q^{-2\rho}f^{\delta}\right) \;=\; w^{\frac{1}{2}}fq^{-\frac{1}{2}}\Theta_{q^4}\left(-q^2f^2\right),$$

with

$$f=\left(\frac{q^8z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{4}}.$$

We have

$$\begin{aligned}
&\mathrm{Tr}_{V(\Lambda_0)+V(\Lambda_1)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_1}_{\lambda_1;1}(zq^{-2})\tilde{\Phi}^{\lambda_1}_{\lambda_1;2}(z)\tilde{\Psi}^{\lambda_1}_{\lambda_{1\pm1};0}(\xi_1)\tilde{\Psi}^{\lambda_{1\pm1}}_{\lambda_1;0}(\xi_2)\right)\\
=\;&\mathrm{Tr}_{\mathcal{F}^a}\left(q^{-2\rho}\hat{\Phi}_1(q^{-2}z)\Phi_2(z)\hat{\Psi}_0(\xi_1)\hat{\Psi}_0(\xi_2)\right)w^{\frac{1}{2}}\mathrm{Tr}_{e^{\frac{\alpha}{2}}\mathbb{C}[Q]}\left(q^{-2\rho}\left(\frac{q^8z^2}{w^2\xi_1\xi_2}\right)^{\frac{\delta}{4}}\right)\\
&\times\mathrm{Tr}_{\mathcal{F}_{\phi^R}}\left(q^{-2\rho}\phi^R(w)\Omega^R_{NS}(q^{-2}\xi_1)\Omega^{NS}_R(q^{-2}\xi_2)\right)\\
=\;&-q^3z(\zeta_1\zeta_2)^{\frac{1}{4}}g^1_{1\pm1}(\xi_1)g^{1\pm1}_1(\xi_2)\frac{q^{-\frac{1}{4}}}{(q^2;q^2)_\infty}\alpha\beta\oint\frac{dw}{2\pi i}\widehat{N}_{12}(z,\xi_1,\xi_2,w)w^{\frac{1}{2}}\left(\frac{q^8z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{4}}\\
&\times\Bigg(G_{\Lambda_+}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I^R_{\Lambda_+}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\\
&\quad-G_{\Lambda_-}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I^R_{\Lambda_-}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\Bigg)\Theta_{q^4}\left(-q^2\left(\frac{q^8z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{2}}\right),
\end{aligned}$$

where we have used the normalisation factors (3.26), (3.28) to write

$$\begin{aligned}
&\tilde{\Phi}^{\lambda_1}_{\lambda_1;1}(zq^{-2})\tilde{\Phi}^{\lambda_1}_{\lambda_1;2}(z)\tilde{\Psi}^{\lambda_1}_{\lambda_{1\pm1};0}(\xi_1)\tilde{\Psi}^{\lambda_{1\pm1}}_{\lambda_1}(\xi_2)\\
=\;&(-q^2z)^{\frac{1}{2}}(-q^4z)^{\frac{1}{2}}g^1_{1\pm1}(\xi_1)g^{1\pm1}_1(\xi_2)\Phi_1(q^{-2}z)\Phi_2(z)\Psi_0(\xi_1)\Psi_0(\xi_2).
\end{aligned}$$

For the second term of the S^+ form-factor, we have

$$\begin{aligned}
&\mathrm{Tr}_{V(\Lambda_0)+V(\Lambda_1)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_1}_{\lambda_1;2}(zq^{-2})\tilde{\Phi}^{\lambda_1}_{\lambda_1;1}(z)\tilde{\Psi}^{\lambda_1}_{\lambda_{1\pm1};0}(\xi_1)\tilde{\Psi}^{\lambda_{1\pm1}}_{\lambda_1;0}(\xi_2)\right)\\
=\;&-q^3z(\zeta_1\zeta_2)^{\frac{1}{4}}g^1_{1\pm1}(\xi_1)g^{1\pm1}_1(\xi_2)\frac{q^{-\frac{1}{4}}}{(q^2;q^2)_\infty}\oint\frac{dw}{2\pi i}\widehat{N}_{21}(z,\xi_1,\xi_2,w)w^{\frac{1}{2}}\left(\frac{q^8z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{4}}\\
&\times\alpha\beta\Bigg(G_{\Lambda_+}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I^R_{\Lambda_+}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\\
&\quad-G_{\Lambda_-}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I^R_{\Lambda_-}(\xi_1^{-\frac{1}{2}},\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\Bigg)\Theta_{q^4}\left(-q^2\left(\frac{q^8z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{2}}\right).
\end{aligned}$$

The remaining integral in each term is manifest through the use of Drinfeld currents $x^\pm(w)$ and so the contour for each is prescribed as in Section 3.3.4, through the consideration of analyticity regions of the normal ordering relations and trace contributions (see Appendices C and D). Using the relations between the vertex operators and their duals (3.30) - (3.33), specifically $\tilde{\Psi}^{\lambda_2}_{\lambda_1,0}(\xi)=-q^{-1}\tilde{\Psi}^{\lambda_2\,*}_{\lambda_1,1}(q^{-2}\xi)$

and $\tilde{\Psi}^{\lambda_1}_{\lambda_2,0}(\xi) = \tilde{\Psi}^{\lambda_2\,*}_{\lambda_1,1}(q^{-2}\xi)$, and the character (3.17), we write

$$\begin{aligned}
&\frac{\langle \mathrm{vac}| S^+ |\xi_1,\xi_2\rangle^{(1;\pm)}_{1,1}}{{}^{(1)}\langle \mathrm{vac}|\mathrm{vac}\rangle^{(1)}} \\
&= \frac{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_1}_{\lambda_1;1}(zq^{-2})\tilde{\Phi}^{\lambda_1}_{\lambda_1;2}(z)\tilde{\Psi}^{*\,\lambda_1}_{\lambda_{1\pm1};1}(\xi_1)\tilde{\Psi}^{*\,\lambda_{1\pm1}}_{\lambda_1;1}(\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\right)} \\
&+\frac{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_1}_{\lambda_1;1}(zq^{-2})\tilde{\Phi}^{\lambda_1}_{\lambda_1;2}(z)\tilde{\Psi}^{*\,\lambda_1}_{\lambda_{1\pm1};1}(\xi_1)\tilde{\Psi}^{*\,\lambda_{1\pm1}}_{\lambda_1;1}(\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\right)} \\
&= -q\frac{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_1}_{\lambda_1;1}(zq^{-2})\tilde{\Phi}^{\lambda_1}_{\lambda_1;2}(z)\tilde{\Psi}^{\lambda_1}_{\lambda_{1\pm1};0}(q^2\xi_1)\tilde{\Psi}^{\lambda_{1\pm1}}_{\lambda_1;0}(q^2\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\right)} \\
&-q\frac{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_1}_{\lambda_1;2}(zq^{-2})\tilde{\Phi}^{\lambda_1}_{\lambda_1;1}(z)\tilde{\Psi}^{\lambda_1}_{\lambda_{1\pm1};0}(q^2\xi_1)\tilde{\Psi}^{\lambda_{1\pm1}}_{\lambda_1;0}(q^2\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\right)},
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{\langle \mathrm{vac}| S^+ |\xi_1,\xi_2\rangle^{(1;\pm)}_{1,1}}{{}^{(1)}\langle \mathrm{vac}|\mathrm{vac}\rangle^{(1)}} \\
&= q^6 z z^{\frac{1}{2}} g^1_{1\pm1}(q^2\xi_1) g^{1\pm1}_1(q^2\xi_2)\frac{q^{-\frac{1}{4}}}{(q^4;q^4)_\infty(-q^2;q^4)_\infty}\alpha\beta \\
&\times\oint\frac{dw}{2\pi i}\left\{\widehat{N}_{12}(z,q^2\xi_1,q^2\xi_2,w)+\widehat{N}_{21}(z,q^{-2}\xi_1,q^{-2}\xi_2,w)\right\} \\
&\times\Bigg(G_{\Lambda_+}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I^R_{\Lambda_+}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}}) \\
&\quad -G_{\Lambda_-}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I^R_{\Lambda_-}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\Bigg) \\
&\times\Theta_{q^4}\left(-\left(\frac{q^8z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{2}}\right).
\end{aligned}$$

4.3.2 The $V(\lambda_0)$ Result

In the case of ground state $V(\lambda_0)$ boundary conditions, we have free field realisation (3.15) and character (3.23). The fermionic trace is taken over the odd Neveu-Schwarz sector as we only have one fermion appearing and so the fermion contribution is from (4.8). For the first term of the S^+ form-factor the bosonic contribution is given in (4.5) and the lattice contribution, recalling Section 3.5.1, is from

$$w^{\frac{1}{2}}\mathrm{Tr}_{e^{\alpha}\mathbb{C}[2Q]}\left(q^{-2\rho}f^{\partial}\right) \quad = \quad f^2 w^{\frac{1}{2}}\Theta_{q^{16}}(-q^{12}f^4),$$

again with

$$f = \left(\frac{q^8 z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{4}}.$$

$$\begin{aligned}
&\frac{\langle \mathrm{vac}|S^+|\xi_1,\xi_2\rangle_{1,1}^{(0;+)}}{{}^{(0)}\langle \mathrm{vac}|\mathrm{vac}\rangle^{(0)}} \\
= \;& \frac{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\,\tilde{\Phi}_{\lambda_2;\,1}^{\lambda_0}(zq^{-2})\tilde{\Phi}_{\lambda_0;\,2}^{\lambda_2}(z)\tilde{\Psi}_{\lambda_1;\,1}^{*\,\lambda_0}(\xi_1)\tilde{\Psi}_{\lambda_0;\,1}^{*\,\lambda_1}(\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right)} \\
& + \frac{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\,\tilde{\Phi}_{\lambda_2;\,2}^{\lambda_0}(zq^{-2})\tilde{\Phi}_{\lambda_0;\,1}^{\lambda_2}(z)\tilde{\Psi}_{\lambda_1;\,1}^{*\,\lambda_0}(\xi_1)\tilde{\Psi}_{\lambda_0;\,1}^{*\,\lambda_1}(\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right)} \\
= \;& -q\frac{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\,\tilde{\Phi}_{\lambda_2;\,1}^{\lambda_0}(zq^{-2})\tilde{\Phi}_{\lambda_0;\,2}^{\lambda_2}(z)\tilde{\Psi}_{\lambda_1;\,0}^{\lambda_0}(q^2\xi_1)\tilde{\Psi}_{\lambda_0;\,0}^{\lambda_1}(q^2\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right)} \\
& -q\frac{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\,\tilde{\Phi}_{\lambda_2;\,2}^{\lambda_0}(zq^{-2})\tilde{\Phi}_{\lambda_0;\,1}^{\lambda_2}(z)\tilde{\Psi}_{\lambda_1;\,0}^{\lambda_0}(q^2\xi_1)\tilde{\Psi}_{\lambda_0;\,0}^{\lambda_1}(q^2\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right)}.
\end{aligned}$$

$$\begin{aligned}
&\frac{\langle \mathrm{vac}|S^+|\xi_1,\xi_2\rangle_{1,1}^{(0;+)}}{{}^{(0)}\langle \mathrm{vac}|\mathrm{vac}\rangle^{(0)}} \\
= \;& -q^2 z(-q^4\xi_1)^{\frac{1}{4}}\frac{q^{-\frac{1}{4}}}{(q^8;q^8)_\infty}\alpha\beta \\
& \times\oint\frac{dw}{2\pi i}\left\{\widehat{N}_{12}(z,q^2\xi_1,q^2\xi_2,w)+\widehat{N}_{21}(z,q^2\xi_1,q^2\xi_2,w)\right\} \\
& \times w^{\frac{1}{2}}\left(\frac{q^4z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{2}}\Bigg(G_{\Lambda_+}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I_{\Lambda_+}^{NS}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}}) \\
& \quad -G_{\Lambda_-}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I_{\Lambda_-}^{NS}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\Bigg) \\
& \times\Theta_{q^{16}}\left(-\frac{q^{16}z^2}{w^2\xi_1\xi_2}\right),
\end{aligned}$$

where we have used the normalisation factor

$$\begin{aligned}&\tilde{\Phi}^{\lambda_0}_{\lambda_2;1}(zq^{-2})\tilde{\Phi}^{\lambda_2}_{\lambda_0;2}(z)\tilde{\Psi}^{\lambda_0}_{\lambda_1;0}(q^2\xi_1)\tilde{\Psi}^{\lambda_1}_{\lambda_0;0}(q^2\xi_2)\\ = \quad & -(-q)^{-1}q^2z(-q^4\xi_1)^{\frac{1}{4}}\Phi_1(zq^{-2})\Phi_2(z)\Psi_0(q^2\xi_1)\Psi_0(q^2\xi_2).\end{aligned}$$

4.3.3 The $V(\lambda_2)$ Result

In the case of ground state $V(\lambda_2)$ boundary condition, we have Fock space free field realisation (3.16) and character (3.25). The fermionic trace is taken over the odd Neveu-Schwarz sector and so the fermion contribution is again from (4.8). The lattice contribution, recalling Section 3.5.1, is from

$$w^{\frac{1}{2}}\mathrm{Tr}_{\mathbb{C}[2Q]}\left(q^{-2\rho}f^{\partial}\right) \quad = \quad w^{\frac{1}{2}}\Theta_{q^{16}}(-q^4f^4),$$

again with

$$f = \left(\frac{q^8z^2}{w^2\xi_1\xi_2}\right)^{\frac{1}{4}}.$$

$$\begin{aligned}&\frac{\langle \mathrm{vac}|S^+|\xi_1,\xi_2\rangle^{(2;-)}_{1,1}}{{}^{(2)}\langle \mathrm{vac}|\mathrm{vac}\rangle^{(2)}}\\
= \quad & \frac{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_2}_{\lambda_0;1}(zq^{-2})\tilde{\Phi}^{\lambda_0}_{\lambda_2;2}(z)\tilde{\Psi}^{*\lambda_0}_{\lambda_1;2}(\xi_1)\tilde{\Psi}^{*\lambda_1}_{\lambda_2;1}(\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right)}\\
& +\frac{\mathrm{Tr}_{V(\lambda_1)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_2}_{\lambda_0;2}(zq^{-2})\tilde{\Phi}^{\lambda_0}_{\lambda_2;1}(z)\tilde{\Psi}^{*\lambda_2}_{\lambda_1;1}(\xi_1)\tilde{\Psi}^{*\lambda_1}_{\lambda_2;1}(\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right)}\\
= \quad & -q\frac{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_0}_{\lambda_2;1}(zq^{-2})\tilde{\Phi}^{\lambda_2}_{\lambda_0;2}(z)\tilde{\Psi}^{\lambda_0}_{\lambda_1;0}(q^2\xi_1)\tilde{\Psi}^{\lambda_1}_{\lambda_0;0}(q^2\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right)}\\
& -q\frac{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\,\tilde{\Phi}^{\lambda_0}_{\lambda_2;2}(zq^{-2})\tilde{\Phi}^{\lambda_2}_{\lambda_0;1}(z)\tilde{\Psi}^{\lambda_0}_{\lambda_1;0}(q^2\xi_1)\tilde{\Psi}^{\lambda_1}_{\lambda_0;0}(q^2\xi_2)\right)}{\mathrm{Tr}_{V(\lambda_0)}\left(q^{-2\rho}\right)}.\end{aligned}$$

$$
\begin{aligned}
&\frac{\langle \mathrm{vac}|S^+|\xi_1,\xi_2\rangle^{(2;-)}_{1,1}}{{}^{(2)}\langle \mathrm{vac}|\mathrm{vac}\rangle^{(2)}} \\
= \quad & -q^6 z(-q^4\xi_2)^{\frac{1}{4}}\frac{q^{-\frac{1}{4}}}{(q^8;q^8)_\infty}\alpha\beta \\
& \times \oint \frac{dw}{2\pi i}\left\{\widehat{N}_{12}(z,q^2\xi_1,q^2\xi_2,w)+\widehat{N}_{21}(z,q^2\xi_1,q^2\xi_2,w)\right\} \\
& \times w^{\frac{1}{2}}\Bigg(G_{\Lambda_+}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I^{NS}_{\Lambda_+}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}}) \\
& \quad -G_{\Lambda_-}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})I^{NS}_{\Lambda_-}(q^{-1}\xi_1^{-\frac{1}{2}},q^{-1}\xi_2^{-\frac{1}{2}},w^{-\frac{1}{2}})\Bigg) \\
& \times \Theta_{q^{16}}\left(-\frac{q^8z^2}{w^2\xi_1\xi_2}\right).
\end{aligned}
$$

With this final expression, we have achieved explicit integral expressions for the two-spinon contribution to the spin-1 S^+ XXZ form-factor with each possible ground state boundary condition.

Chapter 5

The q-Wakimoto Approach

In the previous chapter, we were able to give a general method for the computation of form-factors of the spin-1 XXZ model and a specific integral result for the S^+ form-factor by using careful consideration of the one boson, one fermion free field realisation. This scheme is, however, limited to the level-two, spin-1 case. We now consider an alternative free field realisation which allows us to consider correlation functions and form-factors for arbitrary spin and level. We will introduce a modification of the q-Wakimoto bosonisation used in [62–65]. This is a q-analogue of a classical Wakimoto construction introduced in [100]. Other bosonisations for general level k have been introduced in the literature as q-deformations of the classical Wakimoto case [67, 101–103] and the equivalence between these and our chosen bosonisation [62–64] is discussed in [104].

The q-Wakimoto scheme requires the introduction of objects called screening currents, used in order to ensure that the vertex operators map us between the desired irreducible highest weight modules. Our screening currents differ from the ones used in [62–65], where instead of a contour integral, they take a Jackson integral of the screening current (5.18). The main advantage of our novel definition of the screening currents is in the form of their commutation relations with themselves, the $U_q(\widehat{sl}_2)$ currents and the vertex operators, as will be discussed later on in the chapter. Another advantage is that the current (5.18) has been previously considered in the context of the elliptic RSOS model in [105] and important nilpotency properties of a slight modification to (5.19) that we require to hold, in the framework of BRST cohomology, have been shown to be true. The modification we have used does not affect the proof of the nilpotency.

In order to differentiate between the following free field realisation and the one boson, one fermion scheme, we will now introduce alternative notation for $U_q(\widehat{sl}_2)$. This has another advantage in being consistent with the existing literature for the q-Wakimoto bosonization.

5.1 $U_q(\widehat{sl}_2)$ - Change of Notation

We follow the conventions of [62], [63] and [64] and use the Drinfeld realization of $U_q'(s\hat{l}_2)$ in terms of generators $\{J_n^{\pm}|n \in \mathbb{Z}\}, \{J_n^3|n \in \mathbb{Z}_{\neq 0}\}, \gamma^{\pm 1/2}$ and K with relations

$$\begin{aligned}
[J_n^3, J_m^3] &= \delta_{m+n,0}\frac{1}{n}[2n]\frac{\gamma^n - \gamma^{-n}}{q - q^{-1}}, \\
[J_n^3, K] &= 0, \\
KJ_n^{\pm}K^{-1} &= q^{\pm 2}J_m^{\pm}, \\
[J_n^3, J_m^{\pm}] &= \pm\frac{1}{n}[2n]\gamma^{\pm|n|/2}J_{n+m}^{\pm}, \\
J_{n+1}^{\pm}J_m^{\pm} - q^{\pm 2}J_m^{\pm}J_{n+1} &= q^{\pm 2}J_n^{\pm}J_{m+1}^{\pm} - J_{m+1}^{\pm}J_n^{\pm}, \\
[J_n^+, J_m^-] &= \frac{1}{q - q^{-1}}\left(\gamma^{(n-m)/2}\psi_{n+m} - \gamma^{(m-n)/2}\varphi_{n+m}\right), \\
\gamma^{\pm\frac{1}{2}} &\in \text{the centre of the algebra,}
\end{aligned}$$

where $\{\psi_r, \varphi_s|r, s \in \mathbb{Z}\}$ are related to the generators $\{J_l^3|l \in \mathbb{Z}_{\neq 0}\}$ by the following.

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}\psi_n z^{-n} &= K\exp\left\{(q - q^{-1})\sum_{k=1}^{\infty}J_k^3 z^{-k}\right\}, \\
\sum_{n\in\mathbb{Z}}\varphi_n z^{-n} &= K^{-1}\exp\left\{-(q - q^{-1})\sum_{k=1}^{\infty}J_{-k}^3 z^{k}\right\}.
\end{aligned}$$

As before, the standard Chevalley generators of $U_q(\widehat{sl}_2)$, $\{e_i, f_i, t_i\}$, are given in terms of the Drinfeld generators by the identifications

$$t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = J_0^+, \quad f_1 = J_0^-, \quad e_0 t_1 = J_1^-, \quad t_1^{-1} f_0 = J_{-1}^+.$$

We would like to consider the spin-$\frac{l}{2}$ XXZ model and so require the spin-$\frac{l}{2}$ evaluation representation of $U_q(\widehat{sl}_2)$. For $l \in \mathbb{Z}_{\geq 0}$, we let $V^{(l)}$ denote the $(l+1)$-dimensional $U_q(\widehat{sl}_2)$ module with basis $\left\{ v_m^{(l)} | 0 \leq m \leq l \right\}$ given by

$$\begin{aligned} e_1 v_m^{(l)} &= [m] v_{m-1}^{(l)}, \quad f_1 v_m^{(l)} = [l-m] v_{m+1}^{(l)}, \quad t_1 v_m^{(l)} = q^{l-2m} v_m^{(l)}, \\ e_0 &= f_1, \quad f_0 = e_1, \quad t_0 = t_1^{-1} \quad \text{on} \quad V^{(l)}. \end{aligned}$$

Let $V_z^{(l)}$ be the affinization of $V^{(l)}$, as discussed in Section 2.3.2, i.e. $V_z^{(l)} = V^{(l)} \otimes \mathbb{C}[z, z^{-1}]$, where the action of the Chevalley generators is given by (2.11). Then the action of the Drinfeld generators on $V_z^{(l)}$ is given by

$$\begin{aligned} \gamma^{\pm 1/2} v_m^{(l)} &= v_m^{(l)}, \\ K v_m^{(l)} &= q^{l-2m} v_m^{(l)}, \\ J_n^+ v_m^{(l)} &= z^n q^{n(l-2(m+1))} [m] v_{m-1}^{(l)}, \\ J_n^- v_m^{(l)} &= z^n q^{n(l-2m)} [l-m] v_{m+1}^{(l)} \\ J_n^3 v_m^{(l)} &= \frac{z^n}{n} \left\{ [nl] - q^{n(l+1-m)} (q^n + q^{-n}) [nm] \right\} v_m^{(l)}, \end{aligned}$$

where $v_m^{(l)} = 0$ if $m > l$ or $m < 0$ [62–65].

5.2 Free Field Realisation

We introduce a parameter $k \in \mathbb{Z}_{>0}$ and a set of operators $\{a_n, b_n, c_n, Q_a, Q_b, Q_c \,|\, n \in \mathbb{Z}\}$ satisfying the commutation relations

$$\begin{aligned}
[a_n, a_m] &= \delta_{n+m,0}\frac{[2n][(k+2)n]}{n}, &(5.1)\\
[b_n, b_m] &= -\delta_{n+m,0}\frac{[2n][2n]}{n}, &(5.2)\\
[c_n, c_m] &= \delta_{n+m,0}\frac{[2n][2n]}{n}, &(5.3)\\
[\tilde{a}_0, Q_a] &= 2(k+2), &(5.4)\\
[\tilde{b}_0, Q_b] &= -4, &(5.5)\\
[\tilde{c}_0, Q_c] &= 4, &(5.6)
\end{aligned}$$

where

$$\tilde{x}_0 = \frac{q-q^{-1}}{2\log q}x_0, \quad x \in \{a, b, c\},$$

and all other combinations of operators commute. The value of k will correspond to the level of the $U_q(\widehat{sl}_2)$ highest weight representation we later choose. Let us now introduce three bosonic fields a, b and c. These fields carry parameters $L, M, N \in \mathbb{Z}_{>0}$, $\alpha \in \mathbb{R}$ and are defined by

$$x(L; M, N \,|\, z; \alpha) = -\sum_{n \neq 0} \frac{[Ln]x_n}{[Mn][Nn]} z^{-n} q^{|n|\alpha} + \frac{L\tilde{x}_0}{MN}\log z + \frac{LQ_x}{MN},$$

where again $x \in \{a, b, c\}$. When $L = M$, we abbreviate the notation to

$$\begin{aligned}
x(N \,|\, z; \alpha) &= x(L; L, N \,|\, z; \alpha)\\
&= -\sum_{n \neq 0} \frac{x_n}{[Nn]} z^{-n} q^{|n|\alpha} + \frac{\tilde{x}_0}{N}\log z + \frac{Q_x}{N}.
\end{aligned}$$

We now let $\{a_n, b_n, c_n \,|\, n \in \mathbb{Z}_{\geq 0}\}$ be annihilation operators and $\{a_n, b_n, c_n, Q_a, Q_b, Q_c \,|\, n \in \mathbb{Z}_{<0}\}$ be creation operators. We then realize the $U_q'(\hat{sl}_2)$ currents $J^{\pm}(z)$ and $J^3(z)$

in terms of our fields a, b and c as follows.

$$\begin{aligned}
J^3(z) &= {}_{k+2}\partial_z a(k+2\,|\,q^{-2}z;-1) + {}_2\partial_z b\left(2\,|\,q^{-k-2}z;-\frac{k+2}{2}\right), && (5.7)\\
J^+(z) &= -:\left[{}_1\partial_z \exp\{-c(2\,|\,q^{-k-2}z;0)\}\right] \times \exp\{-b(2\,|\,q^{-k-2}z;-1)\}: && (5.8)\\
J^-(z) &= :\exp\left\{-a\left(k+2\,|\,q^{-2}z;\frac{k+2}{2}\right) + c\left(1;2,k+2\,|\,q^{-k-2}z;0\right)\right\} \\
&\quad \times\left[{}_{k+2}\partial_z \exp\left\{a\left(k+2\,|\,q^{-2}z;-\frac{k+2}{2}\right)\right.\right. \\
&\quad \left.\left. +b\left(2\,|\,q^{-k-2}z;-1\right) + c\left(k+1;2,k+2\,|\,q^{-k-2}z;0\right)\right\}\right]: . && (5.9)
\end{aligned}$$

In the above, ${}_n\partial_z f(z)$, $n \in \mathbb{Z}_{>0}$ denotes the q-difference operator defined by

$$ {}_n\partial_z f(z) \equiv \frac{f(q^n z) - f(q^{-n}z)}{(q-q^{-1})z}. $$

The mode expansions of the currents are set as

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} J^3_n z^{-n-1} &= J^3(z),\\
\sum_{n\in\mathbb{Z}} J^{\pm}_n z^{-n-1} &= J^{\pm}(z).
\end{aligned}$$

It is sometimes convenient in computation to split the currents $J^-(z)$ and $J^+(z)$ into two separate terms as follows:

$$\begin{aligned}
J^-(z) &= \frac{1}{(q-q^{-1})z}\left(J^+_1(z) - J^+_{-1}(z)\right)\\
J^+(z) &= \frac{1}{(q-q^{-1})z}\left(J^-_1(z) - J^-_{-1}(z)\right),
\end{aligned}$$

where

$$
\begin{aligned}
J_1^+(z) &= :\exp\left\{-b\left(2|q^{-k-2}z;1\right) - c\left(2|q^{-k-1}z;0\right)\right\}:, \\
J_{-1}^+(z) &= :\exp\left\{-b\left(2|q^{-k-2}z;1\right) - c\left(2|q^{-k-3}z;0\right)\right\}:, \\
J_1^-(z) &= :\exp\left\{a\left(k+2|q^{k}z;-\frac{k+2}{2}\right) - a\left(k+2|q^{-2}z;\frac{k+2}{2}\right)\right. \\
&\qquad \left. + b\left(2|z;-1\right) + c\left(2|q^{-1}z;0\right)\right\}:, \\
J_{-1}^-(z) &= :\exp\left\{a\left(k+2|q^{-k-4}z;-\frac{k+2}{2}\right) - a\left(k+2|q^{-2}z;\frac{k+2}{2}\right)\right. \\
&\qquad \left. + b\left(2|q^{-2k-4}z;-1\right) + c\left(2|q^{-2k-3}z;0\right)\right\}:.
\end{aligned}
$$

Finally, we let

$$K = q^{\tilde{a}_0+\tilde{b}_0}, \quad \gamma = q^k.$$

We also need the free field realization of the grading operator of the algebra, in this case denoted L_0 in order to differentiate between bosonization schemes. This is realized as

$$
\begin{aligned}
L_0 &= L_0^a + L_0^b + L_0^c, \\
L_0^a &= \frac{\tilde{a}_0(\tilde{a}_0+2)}{4(k+2)} + \sum_{n\geq 1} \frac{n^2}{[2n][(k+2)n]} a_{-n}a_n, \\
L_0^b &= -\frac{\tilde{b}_0(\tilde{b}_0-2)}{8} - \sum_{n\geq 1} \frac{n^2}{[2n]^2} b_{-n}b_n, \\
L_0^c &= \frac{\tilde{c}_0(\tilde{c}_0+2)}{8} + \sum_{n\geq 1} \frac{n^2}{[2n]^2} c_{-n}c_n.
\end{aligned}
$$

5.2.1 Fock Module and the q-Wakimoto module

We now define the Fock module, as in [62], by

$$\begin{aligned} F_{l,s,t} &= \left\{ \prod_{n>0} a_{-n} \prod_{n'>0} b_{-n'} \prod_{n''>0} c_{-n''} \left| l, s, t \right\rangle \right\}, \\ \left| l, s, t \right\rangle &:= \exp\left\{ l \frac{Q_a}{2(k+2)} + \frac{sQ_b + tQ_c}{2} \right\} \left| 0 \right\rangle, \quad (l, s, t \in \mathbb{Z}), \end{aligned} \tag{5.10}$$

where $|0\rangle$ is the vacuum state satisfying

$$a_n \left| 0 \right\rangle = 0, \quad b_n \left| 0 \right\rangle = 0, \quad c_n \left| 0 \right\rangle = 0, \quad n \geq 0.$$

Highest weight vectors $|\lambda_l\rangle$ are given by

$$\left| \lambda_l \right\rangle = \exp\left(\frac{lQ_a}{2(k+2)} \right) \left| 0 \right\rangle. \tag{5.11}$$

The highest weight $U_q(\widehat{sl}_2)$ module $V(\lambda_l)$ of weight λ_l is embedded in the Fock module $\bar{F}_l \equiv \oplus_{s,t\in\mathbb{Z}} F_{l,s,t}$,

$$V(\lambda_l) \hookrightarrow \bar{F}_l.$$

We also introduce conjugate fields

$$\begin{aligned} \eta(z) &= \exp\left\{ c\left(2 \,|\, q^{-k-2} z; 0\right) \right\}, \\ \xi(z) &= \exp\left\{ -c\left(2 \,|\, q^{-k-2} z; 0\right) \right\}. \end{aligned}$$

As discussed in [62] and [63], the Fock space $\bar{F}_l$ contains some redundancies. In the bosonization of our $U_q(\widehat{sl}_2)$ currents (5.7), (5.8) and (5.9), we only use the fields b and c in certain combinations:

$$\begin{aligned} \beta(z) &= -[{}_1\partial_z \xi(z)] : \exp\left\{ -b(2|q^{-k-2} z; -1) \right\} : \\ \gamma(z) &= \eta(z) : \exp\left\{ b(2|q^{-k-2} z; 0) \right\} : . \end{aligned}$$

The field ξ only ever appears within a total difference ${}_n\partial_z$ through β. Taking the contour integral $\oint \frac{dz}{z}$ over ξ within such a z-dependent total difference causes it to vanish when acting within the module because of special properties of its operator product expansions with the $U_q(\widehat{sl}_2)$ currents. As such, we see that the zero mode $\xi_0 = \oint \frac{dz}{z}\xi(z)$ is not contained in the module $V(\lambda_l)$.

The zero mode of the field $\eta(z)$, $\eta_0 \equiv \oint \frac{dz}{2\pi i}\eta(z)$, can be shown to commute with all of the generators of $U_q(\widehat{sl}_2)$, [63] and so $U_q(\widehat{sl}_2)$ acts on the restricted Fock space $\oplus_{s,t\in\mathbb{Z}}\mathrm{Ker}\,\eta_0(F_{l,s,t})$.

We also note that $\tilde{b}_0 + \tilde{c}_0$ acts as zero on the Fock space $F_{l,s,t}$. Because of this, we are able to set the constraint $\tilde{b}_0 + \tilde{c}_0 = 0$ on the Fock space. This is sometimes referred to as *fixing a picture* of the Fock space [63, 67, 101], and takes care of a redundancy in the choice of β, γ vacuum states. This amounts to setting $s = t$ in (5.10) and so from now we will work with the Fock space

$$F_l \equiv \oplus_{s\in\mathbb{Z}} F_{l,s,s} \equiv \oplus_{s\in\mathbb{Z}} F_(l,s). \tag{5.12}$$

The restriction of the Fock space F_l on the kernel of η_0 is called the q-Wakimoto module W_l,

$$W_l = \mathrm{Ker}\,\eta_0(F_l). \tag{5.13}$$

This construction of the q-Wakimoto module, which is detailed in [62] and [63] (and in [67] for a slightly different bosonisation scheme), follows the treatment of the classical case for $\widehat{sl}_2$, [100].

5.3 Elementary Vertex Operators

In order to construct local operators and excited states for the spin-$\frac{l}{2}$ model, we need two types of level k, spin-$\frac{l}{2}$ vertex operators. Let $V(\lambda)$ be a highest weight

$U_q(\widehat{sl}_2)$-module with weight λ, i.e.

$$V(\lambda) := U_q(\widehat{sl}_2)\,|\lambda\rangle\,.$$

Analogously to the spin-1 case, we then have type I vertex operator:

$$\begin{aligned}\Phi_\lambda^{\mu;\,(l)}(z) &= z^{\triangle_\mu-\triangle_\lambda}\tilde{\Phi}_\lambda^{\mu;(l)}(z)\\ \tilde{\Phi}_\lambda^{\mu;\,(l)}(z) &: \quad V(\lambda)\to V(\mu)\otimes V_z^{(l)},\end{aligned}$$

where

$$\lambda_i = (k-i)\Lambda_0 + i\Lambda_1,$$

are dominant integral weights of level k and

$$\triangle_{\lambda_i} = \frac{i(i+2)}{4(k+2)}.$$

The type II vertex operator is as follows:

$$\begin{aligned}\Psi_\lambda^{\mu;\,(l)}(z) &= z^{\triangle_\mu-\triangle_\lambda}\tilde{\Psi}_\lambda^{\mu;(l)}(z)\\ \tilde{\Psi}_\lambda^{\mu;\,(l)}(z) &: \quad V(\lambda)\to V_z^{(l)}\otimes V(\mu).\end{aligned}$$

Splitting the spin-$\frac{l}{2}$ vertex operators into sums of components, we write

$$\begin{aligned}\tilde{\Phi}_\lambda^{\mu;\,(l)}(z) &= \sum_{m=0}^{l}\tilde{\Phi}_{\lambda;\,m}^{\mu;\,(l)}(z)\otimes v_m^{(l)},\\ \tilde{\Psi}_\lambda^{\mu,\,(l)}(z) &= \sum_{m=0}^{l} v_m^{(l)}\otimes\tilde{\Psi}_{\lambda,\,m}^{\mu;\,(l)}(z),\end{aligned}$$

where the type I vertex operators are normalised according to

$$\tilde{\Phi}_\lambda^{\mu;(l)}(z)\,|\lambda\rangle = |\mu\rangle\otimes v_m^{(l)} + \dots\,, \tag{5.14}$$

with $\lambda\equiv\lambda_n = (k-n)\Lambda_0+n\Lambda_1$ and $\mu = n\Lambda_0+(k-n)\Lambda_1 = \lambda_{k-n}$, $n=0,\dots,k$. The vector $v_m^{(l)}$ is determined by setting $m=k-n$. For the type II vertex operators,

we have

$$\tilde{\Psi}_{\lambda_l}^{\lambda_{l\pm1};(1)}(z)\,|\lambda_l\rangle = v_{\mp}^{(1)} \otimes |\lambda_{l\pm1}\rangle + \ldots\,, \tag{5.15}$$

with $l = 0, 1, 2$. In this notation $v_+^{(1)} = v_0^{(1)}$ and $v_-^{(1)} = v_1^{(1)}$. We take the $\ldots$ in the above to mean terms of the form $v \otimes |u\rangle$.

As in the previous cases encountered in Chapter 2 and Chapter 3, we can use the intertwining relations,

$$\begin{aligned}
\tilde{\Phi}_{\lambda}^{\mu;\,(l)}(z) \circ x &= \Delta(x) \circ \tilde{\Phi}_{\lambda}^{\mu;\,(l)}(z),\\
\tilde{\Psi}_{\lambda}^{\mu;\,(l)}(z) \circ x &= \Delta(x) \circ \tilde{\Psi}_{\lambda}^{\mu;\,(l)}(z), \quad \forall x \in U_q(\widehat{sl}_2),
\end{aligned}$$

along with co-multiplication $\Delta : U_q(\widehat{sl}_2) \to U_q(\widehat{sl}_2) \otimes U_q(\widehat{sl}_2)$, in order to derive recursive relations:

$$\begin{aligned}
\tilde{\Phi}_{\lambda,\,m}^{\mu;\,(l)}(z) &= \frac{1}{[l-m]}\left\{\tilde{\Phi}_{\lambda,\,m+1}^{\mu;\,(l)}(z) f_1 - q^{2(m+1)-l} f_1 \tilde{\Phi}_{\lambda,\,m+1}^{\mu;\,(l)}(z)\right\},\\
\tilde{\Psi}_{\lambda,\,m+1}^{\mu;\,(l)}(z) &= \frac{1}{[m]}\left\{\tilde{\Psi}_{\lambda,\,m}^{\mu;\,(l)}(z) e_1 - q^{l-2(m)} e_1 \tilde{\Psi}_{\lambda,\,m}^{\mu;\,(l)}(z)\right\}, \quad m = 0, \ldots, l-1 \quad .
\end{aligned}$$

5.3.1 Type I

For the type I vertex operator, we choose to use the free field realisation of highest component $\tilde{\Phi}_{\lambda,\,l}^{\mu;\,(l)}(z)$ in order to construct the other components. As discussed in [64], the constraints imposed by the intertwining relations on the form of this operator are satisfied if we substitute in the expression

$$\phi_{l,l}(z) =: \exp\left\{a\left(l;\, 2, k+2\,|\,q^k z;\, \frac{k+2}{2}\right)\right\}:$$

for $\tilde{\Phi}_{\lambda,\,l}^{\mu;\,(l)}(z)$. We then use the identification $f_1 = J_0^-$ and

$$J_0^- = \oint \frac{dw}{2\pi i} J^-(w)$$

along with the recursive relation in order to express the other components of the type I vertex operator as a multiple contour integral:

$$\begin{aligned}\phi_{l,m}(z) \;&=\; \frac{1}{[l-m]!}\oint\frac{dw_1}{2\pi i}\oint\frac{dw_2}{2\pi i}\cdots\oint\frac{dw_{l-m}}{2\pi i} \\ &\times[\ldots[[\phi_{l,l}(z),J^-(w_1)]_{q^l},J^-(w_2)]_{q^{l-2}}\ldots J^-(w_{l-m})]_{q^{-l+2(m+1)}},\end{aligned} \tag{5.16}$$

where $m = 0,\ldots l-1$ and $[A,B]_q := AB - qBA$. Borrowing a term from [64], we call these the 'elementary' vertex operators. They are referred to in [62] and [63] as naïve vertex operators.

5.3.2 Type II

For the type II vertex operators, we choose the lowest component of the vertex operator in order to obtain the other components recursively[1]. In [62], we are given the spin-$\frac{l}{2}$ vertex operator of level k,

$$\begin{aligned}\psi_{l,0} \;&=\; :\exp\Big\{a\left(l;2,k+2\,|\,q^{k-2}z;-\frac{k+2}{2}\right) \\ &\qquad +b\left(l;2,1\,|\,q^{-2}z;0\right)+c\left(l;2,1\,|\,q^{-2}z;0\right)\Big\}: \quad ,\end{aligned}$$

which satisfies the simplest intertwining relation for $\tilde{\Psi}^{\mu;\,(l)}_{\lambda,\,0}(z)$. We once again use the recursive relation for components as well as the mode expansion of the current $J^+(z)$ with $J_0^+ = e_1$ in order to obtain

$$\begin{aligned}\psi_{l,m}(z) \;&=\; \frac{1}{[m]!}\oint\frac{du_1}{2\pi i}\oint\frac{du_2}{2\pi i}\cdots\oint\frac{du_m}{2\pi i} \\ &\times[\ldots[[\psi_{l,0}(z),J^+(u_1)]_{q^l},J^+(u_2)]_{q^{l-2}}\ldots J^+(u_m)]_{q^{l-2(m-1)}},\end{aligned} \tag{5.17}$$

for $m = 1,\ldots,l$.

[1] This convention of choosing the highest component to be simple for type I and the lowest for type II appears to have originated in Jimbo and Miwa's original choice for their spin-$\frac{1}{2}$ vertex operators in [23]

5.3.3 Screening Charges

We now consider the spaces on which our naïvely defined vertex operators act. By inspection of its explicit form, we see that our elementary type I vertex operator $\phi_{l,m}(z)$ (5.16) acts as a linear map between q-Wakimoto modules,

$$\phi_{l,m} : W_{l'} \to W_{l'+l}.$$

From this, we see that the naïve construction of elementary vertex operators (5.16) and (5.17) does not provide with a map between all of our desired highest weight modules - they only step us up through the modules and never down. We can fix this by introducing screening currents

$$\begin{aligned} S(z) &= -:\left[{}_1\partial_z \exp\left\{-c\left(2\,|\,q^{-k-2}z;\,0\right)\right\}\right] \\ &\quad \times\exp\left\{-b\left(2\,|\,q^{-k-2}z;\,-1\right) - a\left(k+2\,|\,q^{-2}z;\,-\frac{k+2}{2}\right)\right\} : . \end{aligned} \tag{5.18}$$

By inspection, these operators act on the Fock space $F_{l,s}$ as

$$S(z) : F_{l,s} \to F_{l-2,s-1}.$$

We then introduce our new, modified screening operator[2] Q,

$$Q = \oint_{\mathcal{C}_Q} \frac{dz}{2\pi i z} S(z) \frac{[u - \frac{1}{2} + P_1]_{k+2}}{[u - \frac{1}{2}]_{k+2}} \frac{[1]_{k+2}}{[P_1]_{k+2}},$$

where $z = q^{2u}$ and $[u]_x$ is the theta function [106]

$$[u]_x = \vartheta_1\left(\frac{u}{x}|\tau_x\right),$$

[2]A slightly different form of this Q appears in [105] as a screening charge for the elliptic algebra $U_{q,p}(\widehat{sl}_2)$, which in the limit of a certain parameter r, reverts to the trigonometric case. The screening current $S(z)$ in [105] does not depend on r and so we are able to use the same object in our case.

defined by

$$\vartheta_1(u|\tau) = i\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i(n-1/2)^2\tau}e^{2\pi i(n-1/2)u}.$$

In this particular case, we have $q^{2x} = e^{-2\pi i/\tau_x}$, [105]. At the specific value that we are interested in, $x = k+2$, we use notation $p = q^{2(k+2)}$. The operator P_1 is a special linear combination of zero modes,

$$P_1 = \tilde{a}_0 + \frac{k+2}{2}(\tilde{b}_0 + \tilde{c}_0).$$

This is chosen to ensure that the integrand is single valued in z and the integration is taken over a closed contour $\mathcal{C}_Q$ with poles at $z = qp^m$, $m \geq 0$ lying inside. The theta function $[u]_x$ can be written in alternative notation as

$$\begin{aligned}\vartheta_1\left(\frac{u}{x}|\tau_x\right) &= (-i\tau_x)^{-\frac{1}{2}}q^{\frac{x}{4}}[u]_x \\ &= (-i\tau_x)^{-\frac{1}{2}}q^{\frac{x}{4}}q^{u^2/x-u}\Theta_{q^{2x}}(q^{2u}).\end{aligned}$$

Using this along with the relation $\Theta_p(x) = -x\Theta_p(1/x)$, we express our screening operator in the more convenient form

$$Q = -\oint_{\mathcal{C}_Q}\frac{dz}{2\pi iz}S(z)(q^{-1}z)^{\frac{P_1}{k+2}}q^{\frac{1}{k+2}-1}\frac{\Theta_p(\Pi q/z)\Theta_p(q^2)}{\Theta_p(q/z)\Theta_p(\Pi)}, \tag{5.19}$$

where $\Pi = q^{-2P_1}$.

The use of the contour integral in the definition of Q is rather different compared with the use discussed in the context of $U_q(\widehat{sl}_2)$ currents in Section 2.6. The contour $\mathcal{C}_Q$ is fixed and so when working with Q along with other operators, we really think of this object as a formal sum and, importantly, as a stand alone object. This is in contrast to the contour for the arguments of the Drinfeld currents, which is modified depending on the pole structure generated by its normal ordering relations with other operators. We compute the residue calculation to express Q

as

$$
\begin{aligned}
Q &= \sum_{m\geq 0} S(qp^m)\Pi^m p^{m\frac{P_1}{k+2}} \frac{\Theta_p(q^2)}{(p;p)^3_\infty} q^{\frac{1}{k+2}-1} \\
&= \sum_{m\geq 0} S(qp^m)\frac{\Theta_p(q^2)}{(p;p)^3_\infty} q^{\frac{1}{k+2}-1}.
\end{aligned}
$$

We now introduce the screening charge $Q_n = Q^n$ and use Lemma 4 in [96] in order to write

$$
\begin{aligned}
Q_n &= \frac{[n]_{k+2}!}{n![1]^n_{k+2}} \prod_{j=1}^{n} \left(\oint \frac{dz_j}{2\pi i z_j} S(z_j) \right) \prod_{i<j} \frac{[u_i - u_j]_{k+2}}{[u_i - u_j - 1]_{k+2}} \\
&\times \prod_{i=1}^{n} \frac{[u_i + \frac{1}{2} + P_1 - n]_{k+2}}{[u_i - \frac{1}{2}]} \frac{[1]_{k+2}}{\prod_{j=0}^{n-1} [P_1 - 2j]_{k+2}}.
\end{aligned}
$$

The screening charge acts on the Fock space as

$$
Q_n : F_{l,s} \to F_{l-2n,s-n}.
$$

Now, our highest weight module $V(\lambda_l)$ is embedded in the Fock space $\oplus_{s\in\mathbb{Z}} F_{l,s}$ from which we constructed our q-Wakimoto module W_l and so

$$
S(z) : W_l \to W_{l-2}.
$$

We want our vertex operators to take us from one highest weight module $V(\lambda_n)$ to another highest weight module $V(\lambda_{n'})$, $n, n' \in 0, 1, \ldots, k$, which correspond to q-Wakimoto modules W_{n+1} and $W_{n'+1}$, respectively. Introducing r screening currents $S(t_1), \ldots S(t_r)$ to our vertex operators will take us from W_n to W_{n+l-2r} and so we need r to satisfy $n + l - 2r = n'$ in order for the action of our vertex operators to take us between the desired highest weight modules. The contour integral of the screening current $S(z)$ (i.e. the screening charge) commutes with $U_q(\widehat{sl}_2)$ and η_0[3] and so it is this object that we will use to screen our elementary vertex operators

[3]This is, again, because these operators commute with $S(t)$, at least up to a total difference, and taking a contour integral causes this to vanish.

in the following section, ensuring that the screened vertex operators still act on the space W_l.

5.4 Vertex Operators

5.4.1 Type I

We are now ready to introduce screened vertex operators $\phi^{(r)}_{l,m}$ in terms of the naïve vertex operators $\phi_{l,m}$. We have

$$\tilde{\Phi}^{\lambda_{n'};\,(l)}_{\lambda_n,\,m} \quad = \quad g^{\lambda_{n'};\,(l)}_{\lambda_n}(z)\phi^{(r)}_{l,\,m}(z), \tag{5.20}$$

where $\phi^{(r)}_{l,m}$s is given by

$$\phi^{(r)}_{l,m}(z) = \phi_{l,m}(z)Q_r$$

and $g^{\lambda_{n'};\,(l)}_{\lambda_n}(z)$ is a normalisation factor to be determined through (5.14) The r is fixed by $2r = n + l - n'$, ensuring that we have the right number of screening operators.

5.4.2 Type II

In order to guarantee convergence of the final form of our type II vertex operators, we have to make a slight modification to the screening current. We define $\hat{S}(t)$, by

$$\hat{S}(t) = S(t)\exp\{\tilde{a}_0\log(qp^m)\}, \tag{5.21}$$

and introduce a modified screening operator,

$$\hat{Q} = \oint_{\mathcal{C}_Q}\frac{dz}{2\pi iz}\hat{S}(z)\frac{[u-\frac{1}{2}+P_1]_{k+2}}{[u-\frac{1}{2}]_{k+2}}\frac{[1]_{k+2}}{[P_1]_{k+2}}. \tag{5.22}$$

The contour is the same one as introduced for Q, $\mathcal{C}_Q$ with poles at $z = qp^m$, $m \geq 0$. Again considering this as a formal sum, we have

$$\hat{Q} = \sum_{m \geq 0} S(qp^m) \exp\{\tilde{a}_0 \log(qp^m)\} \frac{\Theta_p(q^2)}{(p;p)^3_\infty} q^{\frac{1}{k+2}-1}.$$

The modification (5.21) to the screening operator $S(t)$ preserves the single-valuedness of the integrand of the screening operator, but also guarantees convergence of the normalisation factor defined as above, whilst not changing the sectors between which it maps. The modified screening operator $\hat{Q}$ still commutes with the $U_q(\widehat{sl}_2)$ currents and the zero mode η_0, ensuring that the screened type II vertex operators act on the correct space.

We can now introduce our complete type II vertex operators as

$$\tilde{\Psi}^{\lambda_{n'};(l)}_{\lambda_n, m} \quad = \quad \bar{g}^{\lambda_{n'};(l)}_{\lambda_n}(z)\psi^{(r)}_{l,m}(z), \tag{5.23}$$

where

$$\psi^{(r)}_{l,m}(z) = \psi_{l,m}\hat{Q}_r.$$

The pre-factor $\bar{g}^{\lambda_{n'};(l)}_{\lambda_n}$ is the type II normalisation factor arising through demanding (5.15).

Once attached to a vertex operator, the contour of the integrals within our screening operators (5.19) and (5.22) are defined by the usual contour outlined in the previous section (i.e. with $z = qp^m$ inside).

If we have vertex operators that are screened by multiple charges (as we will encounter when computing correlation functions or form-factors), each of these is considered as a stand alone object with fixed contour $\mathcal{C}_Q$ so that the formal residue sum is taken before consideration of any poles that would be generated through the normal ordering of a pair of screening currents (5.18).

5.5 Type I Normalisation

In this section, we discuss the computation of the type I normalisation factors arising through relations (5.14) and (5.20). The number of screening charges, r, required for a vertex operator acting between λ_n and $\lambda_{n'}$ was previously defined by $2r = n + l - n'$. For perfect vertex operators, where $l = k$ and so where $n' = l - n$, we have $r = n$. As an example of the computation of the normalisation factors, the simplest case to look at is when $n = 0$ since we will get no screening charges. We have

$$\tilde{\Phi}_{\lambda_0}^{\lambda_l;(l)}(z) = \sum_{m=0}^{l} \tilde{\Phi}_{\lambda_0,m}^{\lambda_l;(l)} \otimes v_m,$$

and want to look at the leading term

$$\begin{aligned} \tilde{\Phi}_{\lambda_0}^{\lambda_l;(l)}(z)\,|\lambda_0\rangle &= g_{\lambda_0}^{\lambda_l;(l)}(z)\phi_{l,l}(z)\,|\lambda_0\rangle \otimes v_l + \ldots \\ &= |\lambda_l\rangle \otimes v_l + \ldots, \end{aligned}$$

in order to define normalisation factor $g_{\lambda_0}^{\lambda_l;(l)}(z)\phi_{l,l}(z)$. Using the explicit free field realisation of $\phi_{l,l}(z)$ and highest weight state $|\lambda_l\rangle$, we have

$$\begin{aligned} &:\exp\left\{a\left(l;\,2,k+2\,|\,q^k z;\,\frac{k+2}{2}\right)\right\}:|\lambda_0\rangle \\ &= \exp\Bigg\{-\sum_{n\neq 0}\frac{a_n\,[ln]}{[2n][(k+2)n]}(q^k z)^{-n} q^{\frac{k+2}{2}|n|} \\ &\qquad +\frac{l\,\tilde{a}_0}{2(k+2)}\log(q^2 z) + \frac{l\,Q_a}{2(k+2)}\Bigg\}\,|\lambda_0\rangle \\ &= \exp\left\{\frac{l\,Q_a}{2(k+2)}\right\}|\lambda_0\rangle + \ldots \\ &= |\lambda_l\rangle + \ldots \end{aligned}$$

and find that

$$g_{\lambda_0}^{\lambda_l;(l)}(z) = 1.$$

The number of screening charges will increase with n and the normalisation factors will become increasingly complicated, but can be computed using the same method as above.

5.6 Type II Normalisation

For the the normalisation of the type II vertex operators, we use (5.15) and (5.23). We also have

$$\tilde{\Psi}_{\lambda}^{\mu,\,(1)}(z) \;=\; \sum_{m=0}^{1} v_m^{(1)} \otimes \tilde{\Psi}_{\lambda,\,m}^{\mu;\,(1)}(z).$$

We now compute normalisation factors $\bar{g}_{\lambda_n}^{\lambda_{n'};\,(1)}(z)$, where

$$\tilde{\Psi}_{\lambda_n,\,m}^{\lambda_{n'};\,(1)} \;=\; \bar{g}_{\lambda_n}^{\lambda_{n'};\,(1)}(z)\psi_{1,\,m}^{(r)}(z).$$

Choosing to look at $\tilde{\Psi}_{\lambda}^{\mu,\,(1)}(z)$ with $\lambda = \lambda_l$ and $\mu = \lambda_{l+1}$, we have $r = 0$ in (5.23) and so obtain

$$\tilde{\Psi}_{\lambda_l,m}^{\lambda_{l+1};(1)}(z) = \bar{g}_{\lambda_l}^{\lambda_{l+1};(1)}(z)\psi_{1,m}(z).$$

Taking, instead, λ_l and λ_{l-1}, we have $r = 1$ and find that

$$\begin{aligned}
\tilde{\Psi}_{\lambda_l,m}^{\lambda_{l-1};(1)}(z) &= g_{\lambda_l}^{\lambda_{l-1};(1)}(z)\psi_{1,m}^{(1)}(z) \\
&= g_{\lambda_l}^{\lambda_{l-1};(1)}(z)\psi_{1,m}(z)\tilde{Q}.
\end{aligned}$$

5.6.1 Normalisation factor $g_{\lambda_l}^{\lambda_{l+1}}(z)$

Everything here works in the same way as in [62] as there are no screening currents required and it is within the definition of screening operators that our set up differs.

From the above, we have

$$\begin{aligned}\tilde{\Psi}_{\lambda_l}^{\lambda_{l+1}}(z)\,|\lambda_l\rangle &= \bar{g}_{\lambda_l}^{\lambda_{l+1}}(z)\,(v_0\otimes\psi_{1,0}+v_1\otimes\psi_{1,1})\,|\lambda_l\rangle \\ &= v_1\otimes|\lambda_{l+1}\rangle+\ldots \qquad .\end{aligned}$$

We recall the definition of the highest weight vector

$$|\lambda_l\rangle=\exp\left(\frac{lQ_a}{2(k+2)}\right)|0\rangle\,,$$

along with

$$\begin{aligned}\psi_{l,0} &= :\exp\Big\{a\left(l;2,k+2\,|\,q^{k-2}z;-\frac{k+2}{2}\right) \\ &\qquad +b\left(l;2,1\,|\,q^{-2}z;0\right)+c\left(l;2,1\,|\,q^{-2}z;0\right)\Big\}: \quad ,\end{aligned}$$

and

$$\begin{aligned}&a\left(1;2,k+2\,|\,q^{k-2}z;-\frac{k+2}{2}\right) \\ =&\ -\sum_{n\neq0}\frac{[n]a_n}{[2n][n(k+2)]}(q^{k-2}z)^{-n}q^{-\frac{k+2}{2}|n|}+\frac{\tilde{a}_0}{2(k+2)}\log(q^{k-2}z)+\frac{Q_a}{2(k+2)}.\end{aligned}$$

We need to look at the action $\psi_{1,1}(z)\,|\lambda_l\rangle$ and so use

$$\begin{aligned}&\psi_{1,1}(z) \\ =&\ \oint\frac{du}{2\pi i}\left(\psi_{1,0}(z)J^+(u)-qJ^+(u)\psi_{1,0}(z)\right) \\ =&\ -\frac{1}{q-q^{-1}}\oint\frac{du}{2\pi i}\frac{1}{u}\left(\psi_{1,0}(z)\left[J_1^+(u)-J_{-1}^+(u))\right]-q\left[J_1^+(u)-J_{-1}^+(u)\right]\psi_{1,0}(z)\right) \\ =&\ -\oint\frac{du}{2\pi i}\frac{1}{u}\left(\frac{q^kz}{u-q^{k-1}z}:\psi_{1,0}(z)J_1^+(u):+\frac{qz}{z-q^{-1-k}u}:\psi_{1,0}(z)J_{-1}^+(u):\right),\end{aligned}$$

where we use normal ordering factors listed in Appendix G.

The action of the zero modes on $|\lambda_l\rangle$ is computed as

$$\begin{aligned} & \exp\left(\frac{Q_a}{2(k+2)}\right)\exp\left(\frac{\tilde{a}_0}{2(k+2)}\log(q^{k-2}z)\right)\exp\left(\frac{lQ_a}{2(k+2)}\right)|0\rangle \\ = & \exp\left(\frac{l\log(q^{k-2}z)}{2(k+2)}\right)\exp\left(\frac{(l+1)Q_a}{2(k+2)}\right)\exp\left(\frac{\tilde{a}_0}{2(k+2)}\log(q^{k-2}z)\right)|0\rangle \\ = & \left(q^{k-2}z\right)^{l/2(k+2)}|\lambda_{l+1}\rangle\,. \end{aligned}$$

The contributions of the zero modes Q_b and Q_c cancel as we have $\exp\left(\frac{Q_b}{2}+\frac{Q_c}{2}\right)$ appearing in $\Psi_{1,0}(z)$ and $\exp\left(-\frac{Q_b}{2}-\frac{Q_c}{2}\right)$ appearing in $J_+(u)$. Even without this particular combinations, the b and c creation zero modes, when used in combination, satisfy $\exp\{Q_b+Q_c\}\,|\lambda_l\rangle = 1$.

Putting all of this together, $\psi_{1,1}(z)$ acts on λ_l as

$$\begin{aligned} & \Psi_{1,1}(z)\,|\lambda_l\rangle \\ = & \; qz\oint\frac{du}{2\pi i}\frac{1}{u}\left(\frac{1}{z-q^{1-k}u}:\Psi_{1,0}(z)J_1^+(u):+\frac{1}{z-q^{-1-k}u}:\Psi_{1,0}(z)J_{-1}^+(u):\right)|\lambda_l\rangle \\ = & \;(q^{k-2}z)^{\frac{l}{2(k+2)}}\oint_{\mathcal{C}}\frac{du}{2\pi i}\frac{1}{u}\left(\frac{qz}{z-q^{1-k}u}+\frac{qz}{z-q^{-1-k}u}\right)|\lambda_{l+1}\rangle \\ = & \; q(q^{k-2}z)^{\frac{l}{2(k+2)}}\,|\lambda_{l+1}\rangle\,. \end{aligned}$$

Taking the analyticity regions into account, we have chosen the contour $\mathcal{C}$ such that poles at $u=q^{k-1}z$ and $u=0$ are inside and $u=q^{k+1}z$ is outside. The result is obtained by taking the sum of these residues.

5.6.2 Normalisation factor $g_{\lambda_l}^{\lambda_{l-1}}(z)$

In this case, we have

$$g_{\lambda_l}^{\lambda_{l-1}}(z)\left(v_0\otimes\psi_{1,0}(z)\tilde{Q}+v_1\otimes\ldots\right)|\lambda_l\rangle = v_0\otimes|\lambda_{l-1}\rangle+\ldots \qquad ,$$

and so we want to look at the action

$$\psi_{1,0}(z)\tilde{Q}\;|\lambda_l\rangle\,.$$

We use the normal ordering factors

$$\begin{aligned}
\psi_{1,0}(z)S_1(t) &= (q^{k-2}z)^{-1/k+2}\frac{(q^{-k+1}pt/z;p)_\infty}{(q^{-k-1}pt/z;p)_\infty} : S_1(t)\psi_{1,0}(z) : \\
\psi_{1,0}(z)S_{-1}(t) &= (q^{k-2}z)^{-1/k+2}\frac{(q^{-k+1}t/z;p)_\infty}{(q^{-k-1}t/z;p)_\infty} : S_{-1}(t)\psi_{1,0}(t) : ,
\end{aligned}$$

where $p = q^{2(k+2)}$ and we split $S(t)$ as

$$S(z) = -\frac{1}{(q-q^{-1})z}\sum_{\delta=\pm 1}\delta S_\delta(z),$$

$$\begin{aligned}
S_\delta(z) &= : \exp\Bigg\{ -a\left(k+2|q^{-2}z;-\frac{k+2}{2}\right) \\
&\quad -b\left(2|q^{-k-2}z;-1\right) - c\left(2|q^{-k-2+\delta}z;0\right)\Bigg\}.
\end{aligned}$$

For the action of $: S_{-1}(t)\psi_{1,0}(z) :$, we focus on the contribution of normal ordered zero modes from the bosonic a fields and are looking at

$$\begin{aligned}
& e^{\frac{Q_a}{2(k+2)}-\frac{Q_a}{k+2}} e^{\frac{\tilde{a}_0}{k+2}\left(1/2\log(q^{k-2}z)-\log(q^{-2}t)\right)} |\lambda_l\rangle \\
&= e^{\frac{Q_a}{2(k+2)}-\frac{Q_a}{k+2}} e^{\frac{\tilde{a}_0}{k+2}\left(1/2\log(q^{k-2}z)-\log(q^{-2}t)\right)} e^{\frac{lQ_a}{2(k+2)}} |0\rangle \\
&= \left(\frac{q^{k+2}z}{t^2}\right)^{\frac{l}{2(k+2)}} e^{\frac{(l-1)Q_a}{2(k+2)}} e^{\frac{\tilde{a}_0}{k+2}\left(1/2\log(q^{k-2}z)-\log(q^{-2}t)\right)} |0\rangle \\
&= \left(\frac{q^{k+2}z}{t^2}\right)^{\frac{l}{2(k+2)}} |\lambda_{l-1}\rangle ,
\end{aligned}$$

The contributions of the zero modes Q_b and Q_c cancel once again. The action of $S_1(t)$ is exactly the same as only the a-field contributes and is the same for each.

The issue with the formulation of type II vertex operators using screening current

(5.18) is that we encounter the following divergent contribution in the normalisation factor:

$$\begin{aligned}
\sum_{m\geq 0}\frac{(q^{-k+2}p^{m+1}/z;p)_\infty}{(q^{-k}p^m/z;p)_\infty}q^{-2ml} &= \frac{(q^{-k+2}p/z;p)_\infty}{(q^{-k}/z;p)_\infty}\sum_{m\geq 0}\frac{(q^{-k}/z;p)_m}{(q^{-k+2}p/z;p)_m}\frac{(p;p)_m}{(p;p)_m}q^{-2ml} \\
&= \frac{(q^{-k+2}p/z;p)_\infty}{(q^{-k}/z;p)_\infty}{}_2\phi_1\left[\begin{matrix} q^{-k}/z,p \\ q^{-k+2}p/z\end{matrix};p,q^{-2l}\right].
\end{aligned}$$

Noting that

$$\exp\{\tilde{a}_0\log(t)\}\left|\lambda_l\right\rangle = t^l\left|\lambda_l\right\rangle,$$

the calculation of the normalisation factor with modified screening current (5.21) now gives us

$$\begin{aligned}
&\psi_{1,0}(z)\tilde{Q}\left|\lambda_l\right\rangle \\
=\ & -\psi_{1,0}(z)\sum_{m\geq 0}S(qp^m)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}-1}\exp\{\tilde{a}_0\log(qp^m)\}\left|\lambda_l\right\rangle \\
=\ & \frac{(q^kz)^{\frac{l}{2(k+2)}}(q^{k-3}z)^{-\frac{1}{k+2}}}{zq^{k-l+1}}\frac{\Theta_p(q^2)}{(p;p)^2_\infty}\sum_{m\geq 0}\frac{(q^{-k+2}p^{m+1}/z;p)_\infty}{(q^{-k}p^m/z;p)_\infty}q^{2(k+1)ml}\left|\lambda_{l-1}\right\rangle+\ldots.
\end{aligned}$$

We can then write

$$\begin{aligned}
\sum_{m\geq 0}\frac{(q^{-k+2}p^{m+1}/z;p)_\infty}{(q^{-k}p^m/z;p)_\infty}q^{2(k+1)ml} &= \frac{(q^{-k+2}p/z;p)_\infty}{(q^{-k}/z;p)_\infty}\sum_{m\geq 0}\frac{(q^{-k}/z;p)_m}{(q^{-k+2}p/z;p)_m}\frac{(p;p)_m}{(p;p)_m}q^{2(k+1)ml} \\
&= \frac{(q^{-k+2}p/z;p)_\infty}{(q^{-k}/z;p)_\infty}{}_2\phi_1\left[\begin{matrix} q^{-k}/z,p \\ q^{-k+2}p/z\end{matrix};p,q^{2(k+1)l}\right]
\end{aligned}$$

and use Heine's transformation formula [98] and the q-binomial theorem [99] to simplify

$${}_2\phi_1\left[\begin{matrix} q^{-k}/z,p \\ q^{-k+2}p/z\end{matrix};p,q^{2(k+1)l}\right] = 1$$

and so

$$\psi_{1,0}(z)\tilde{Q}\,|\lambda_l\rangle \;=\; \frac{(q^k z)^{\frac{l}{2(k+2)}}(q^{k-3}z)^{-\frac{1}{k+2}}}{zq^{k-l+1}}\frac{\Theta_p(q^2)}{(p;p)_\infty^2}\frac{(q^{-k+2}p/z;p)_\infty}{(q^{-k}/z;p)_\infty}\,|\lambda_{l-1}\rangle + \ldots,$$

meaning that

$$g_{\lambda_l}^{\lambda_{l-1}}(z) = \left(\frac{(q^k z)^{\frac{l}{2(k+2)}}(q^{k-3}z)^{-\frac{1}{k+2}}}{zq^{k-l+1}}\frac{\Theta_p(q^2)}{(p;p)_\infty^2}\frac{(q^{-k+2}p/z;p)_\infty}{(q^{-k}/z;p)_\infty}\right)^{-1}.$$

5.7 BRST Cohomology

The Fock modules (5.12) and (5.13) that were constructed in Section 5.2.1 are reducible for certain values of l because of the existence of singular vectors. In order to compute form-factors, we need a way of extracting the irreducible part over which we would like to take a trace.

Again following [62, 63], we introduce the following notation for the labelling of the q-Wakimoto module (5.13).

$$W_{n,n'} \equiv W_{l_{n,n'}}; \quad l_{n,n'} = n - \frac{n'P}{P'} - 1, \tag{5.24}$$

where (P, P') are co-prime integers such that $\frac{P}{P'} = k+2$. We introduce this because in the case where $1 \le n \le P-1$ and $0 \le n' \le P'-1$, the q-Wakimoto module $W_{n,n'}$is reducible [86] and the resolution of the irreducible highest weight module $V(\Lambda_l)$ of $U_q(\hat{sl}_2)$ can be given in terms of a cohomology group in a complex of q-Wakimoto modules using a q-analogue of Bernard-Felder BRST cohomology [92, 93].

We recall the previously defined screening charge $Q_n = Q^n$ for the type I vertex operators, see (5.19), which acts on q-Wakimoto module $W_{m,m'}$ as

$$Q_n : W_{m,m'} \to W_{m-2n,m'}.$$

The charge Q_n satisfies all the properties required to be used as the charge in the BRST cohomology approach and so we also call it our BRST operator. The properties are as follows:

1. Q_n commutes with $U_q(\hat{sl_2})$ and the zero mode η_0.
2. We have nilpotency relations $Q_n Q_{P-n} = Q_{P-n} Q_n = 0$.

In this case, the sequence

$$\ldots \xrightarrow{Q_n} W_{-n+2P,n'} \xrightarrow{Q_{P-n}} W_{n,n'} \xrightarrow{Q_n} W_{-n,n'} \xrightarrow{Q_{P-n}} W_{n-2P,n'} \xrightarrow{Q_n} \ldots \quad , \tag{5.25}$$

is a complex with

$$\mathrm{Im} Q_n^{[s-1]} \subset \mathrm{Ker} Q_n^{[s]},$$

where $Q_n^{[2\alpha]} = Q_n$ and $Q_n^{[2\alpha-1]} = Q_{P-n}$ $(\alpha \in \mathbb{Z})$.

As discussed previously, the nilpotency property for this choice of BRST operator is proven in [105]. The relation between our operator Q and the operator, call it $\tilde{Q}$, in [105] is

$$Q^n = \tilde{Q}^n \frac{[1]^n}{\prod_{j=0}^{n-1} [P_1 - 2j]_{k+2}},$$

and this substitution in does not affect the proof of the nilpotency relation appearing in [105]. This is one of our reasons for choosing to use a different BRST charge from the one appearing in [63][4].

The complex (5.25) has cohomology groups

$$\mathrm{Ker}\, Q_n^{[s]} / \mathrm{Im}\, Q_n^{[s-1]} = \begin{cases} 0, & s \neq 0 \\ \mathcal{H}_{n,n'}, & s = 0 \end{cases}.$$

[4]There are some issues with the proof of the nilpotency relations for the BRST charge used in [63]. In particular the relation $\mathcal{A}_s^P = 1$, stated underneath eq. (43) in that paper can be shown not to hold through a simple expansion to high order in powers of q.

The only non trivial cohomology group, $\mathcal{H}_{n,n'}$, is the irreducible highest weight module for $U_q(\hat{sl}_2)$. It has highest weight $\lambda_{l_{n,n'}}$.

As a corollary, the trace of an operator $\mathcal{O}$ (which will eventually be a product of vertex operators) over $\mathcal{H}_{n,n'} \equiv \mathcal{H}_{\lambda_{l_{n,n'}}}$ is given in [63] in terms of an alternating sum of traces over the q-Wakimoto module:

$$\mathrm{Tr}_{\mathcal{H}_{n,n'}} \mathcal{O} = \sum_{s \in \mathbb{Z}} (-1)^s \mathrm{Tr}_{W^{[s]}_{n,n'}} \mathcal{O}^{[s]}, \tag{5.26}$$

where $\mathcal{O}^{[s]}$ is a graded physical operator defined by the recursive relations

$$Q^{[s]}_n \mathcal{O}^{[s]} = \mathcal{O}^{[s+1]} Q^{[s]}_n, \quad \mathcal{O}^{[0]} = \mathcal{O}. \tag{5.27}$$

The graded q-Wakiomoto module is understood to be the module at level $[s]$ in the complex (5.25), with $W^{[0]}_{n,n'} \equiv W_{n,n'}$. The final step is to then relate these traces over q-Wakimoto modules to traces over Fock modules through the definition of the former as restrictions of the latter.

Regarding the Fock space $F_{l,s}$ as a $\xi - \eta$ module, we can write it as

$$F_{l,s} = F^0 + \xi_0 F_0, \quad F^0 = \mathrm{Ker}(\eta_0).$$

As discussed in the definition of the Fock and q-Wakimoto modules in Section 5.2.1, the $\beta - \gamma$ system in terms of which we bosonize $U_q(\widehat{sl}_2)$ in the Wakimoto representation is independent of ξ_0. Therefore, to get the space isomorphic to the Wakimoto module from our Fock space $F_{l,s}$, we need to 'pick up' F_0 and 'throw out' $\xi_0 F^0$. We can do this by inserting zero modes ξ_0 and η_0 into the trace over $F_{l,s}$. The zero modes satisfy anti-commutation relation $\{\xi_0, \eta_0\} = 1$ and so we can write[5]

$$\begin{aligned} \mathrm{Tr}_{F_{l,s}} (\xi_0 \eta_0 \mathcal{O}) &= \mathrm{Tr}_{F^0 + \xi_0 F^0} (\mathcal{O}) - \mathrm{Tr}_{F^0 + \xi_0 F^0} (\eta_0 \xi_0 \mathcal{O}) \\ &= \mathrm{Tr}_{F^0} (\mathcal{O}) . \end{aligned}$$

[5]The author is grateful to H. Konno for discussion on this issue.

Combining this with the picture fixing demand that $\tilde{b}_0 + \tilde{c}_0 = 0$ on $F_{l,s}$, the final trace expression (as in [62, 63]) is

$$\mathrm{Tr}_{W^{[s]}_{n,n'}} \mathcal{O}^{[s]} = \mathrm{Tr}_{F^{[s]}_{n,n'}} \left(\oint \frac{dw_0}{2\pi i w_0} \xi(w_0) \oint \frac{dw}{2\pi i} \eta(w) \mathcal{O}^{[s]} \right) \Bigg|_{\tilde{b}_0+\tilde{c}_0=0} . \tag{5.28}$$

The graded Fock $F^{[s]}_{n,n'}$ space is, again, understood to be the module at level $[s]$ in the complex (5.25), with associated q-Wakimoto module $W^{[s]}_{n,n'}$ and $F^{[0]}_{n,n'} \equiv F_{n,n'}$.

With this set up we, in theory, have everything we need in order to express correlation functions and form-factors (3.7), (3.8), as traces of q-Wakimoto bosonized vertex operators (5.20), (5.23). The individual traces we have to take will be relatively simple compared with those in Chapters 3 and 4 there are no fermion contributions to consider and we can use the bosonic trace formula (3.35) for each of the three types of boson in the q-Wakimoto construction.

The complicated part of taking traces in this realisation is that we have an infinite alternating sum of traces over *graded* operators defined by BRST relations (5.27). In the next section, we will consider the form of these relations in some specific cases and consider the feasibility of extracting form-factor expressions from trace expressions of the form (5.26).

5.8 Level-two BRST Operator Relations

For simplicity, we return to the spin-1 case by setting $l = 2$ and consider the BRST relations for an operator $\mathcal{O}$ built from spin-1, type I and spin-$\frac{1}{2}$, type II vertex operators. From this, we will be able to investigate the form of the traces that contribute to the two-particle spin-1 form-factors.

As discussed in the previous section, the BRST operator we use is the same as the type I screening operator (5.19). In the level-two case, we have $P = 4$ and $P' = 1$ in (5.24). This means that when taking the trace over $\mathcal{H}_{\lambda_0} \equiv V(\lambda_0)$, the BRST operator (5.27) relations for $s \geq 0$ are

$$\begin{aligned} Q_1\mathcal{O}^{[0]} &= \mathcal{O}^{[1]}Q_1 \\ Q_3\mathcal{O}^{[1]} &= \mathcal{O}^{[2]}Q_3 \\ Q_1\mathcal{O}^{[2]} &= \mathcal{O}^{[3]}Q_1 \\ Q_3\mathcal{O}^{[3]} &= \mathcal{O}^{[4]}Q_3 \\ &\vdots \end{aligned} \tag{5.29}$$

For $\mathcal{H}_{\lambda_1} \equiv V(\lambda_1)$, however, we have repeated relation

$$Q_2\mathcal{O}^{[s]} = \mathcal{O}^{[s+1]}Q_2, \tag{5.30}$$

and for $\mathcal{H}_{\lambda_2} \equiv V(\lambda_2)$, we have alternating relations

$$\begin{aligned} Q_3\mathcal{O}^{[0]} &= \mathcal{O}^{[1]}Q_3 \\ Q_1\mathcal{O}^{[1]} &= \mathcal{O}^{[2]}Q_1 \\ Q_3\mathcal{O}^{[2]} &= \mathcal{O}^{[3]}Q_3 \\ Q_1\mathcal{O}^{[3]} &= \mathcal{O}^{[4]}Q_1 \\ &\vdots \quad . \end{aligned} \tag{5.31}$$

The graded operators $\mathcal{O}^{[s]}$ are specified by the commutation relations of the zero-level operator $\mathcal{O}^{[0]} = \mathcal{O}$ with the BRST operator Q, (5.19). For the majority of the operators that we use, these relations have a special form which makes their BRST relations relatively simple. We will discuss this, and the shortcomings of our modified screening operator (5.22), in the next section.

5.8.1 Pseudo-Constants

The commutation relations of the simple vertex operators with the BRST operator are special because of the following proposition.

Proposition 5.1. *The simple vertex operator components $\phi_{2,2}(z)$, $\psi_{1,0}(\xi)$ and the screening current $S(t)$ satisfy the following commutation relations*

$$\begin{aligned}
S(t_1)S(t_2) &= \mathcal{A}_S(t_1/t_2)S(t_2)S(t_1) \\
S(t)\phi_{2,2}(z) &= \mathcal{A}_{S\phi}(t/z)\phi_{2,2}(z)S(t) \\
S(t)\psi_{1,0}(\xi) &= \mathcal{A}_{S\psi}(t/\xi)\psi_{1,0}(\xi)S(t),
\end{aligned}$$

where all of the $\mathcal{A}$ factors are p-pseudo-constants. This means that

$$\mathcal{A}_{SA}(t) = \mathcal{A}_{SA}(pt), \quad p = q^{2(k+2)}$$

for $A \in S, \Psi, \Phi$.

We prove Proposition 5.1 by calculating the $\mathcal{A}_{SA}$ factors explicitly, using the normal ordering relations detailed in Appendix G. Firstly, considering $\mathcal{A}_{S\phi}$, we use

$$\begin{aligned}
S(t)\phi_{2,2}(z) &= (q^{-2}t)^{-\frac{2}{k+2}}\frac{(q^2pz/t;p)_\infty}{(q^{-2}pz/t;p)_\infty} :S(t)\phi_{2,2}(z):, \quad |t| > q^{-2}p|z| \\
\phi_{2,2}(z)S(t) &= (q^k z)^{-\frac{2}{k+2}}\frac{(q^2t/z;p)_\infty}{(q^{-2}t/z;p)_\infty} :S(t)\phi_{2,2}(z):, \quad |z| > q^{-2}|t|,
\end{aligned}$$

to obtain

$$\mathcal{A}_{S\phi}(t/z) = q^2\frac{\Theta_p(q^{-2}t/z)}{\Theta_p(q^2t/z)}\left(\frac{z}{t}\right)^{\frac{2}{k+2}}.$$

Using the relation

$$\Theta_p(pz) = -z^{-1}\Theta_p(z),$$

it is then easy to show that $\mathcal{A}_{S\phi}(z) = \mathcal{A}_{S\phi}(p^n z)$, $n \in \mathbb{Z}$.

We next consider $\mathcal{A}_{S\psi}$ and start with the normal ordering relations

$$\begin{aligned}
S(t)\psi_{1,0}(z) &= -\frac{(q^{-2}t)^{-\frac{1}{k+2}}}{(q-q^{-1})t}\frac{(q^{k+1}z/t;p)_\infty}{(q^{k-1}z/t;p)_\infty}\big(q^{-1}:S_1(t)\psi_{1,0}(z): \\
&\quad -q\frac{(1-q^{k-1}z/t)}{(1-q^{k+1}z/t)}:S_{-1}(t)\psi_{1,0}(z):\big) \\
&= -\frac{(q^{-2}t)^{-\frac{1}{k+2}}}{(q-q^{-1})t}\frac{(q^{k+1}pz/t;p)_\infty}{(q^{k-1}z/t;p)_\infty}\big(q^{-1}(1-q^{k+1}z/t):S_1(t)\psi_{1,0}(z): \\
&\qquad -q(1-q^{k-1}z/t):S_{-1}(t)\psi_{1,0}(z):\big), \quad |t|>q^{k-1}|z| \\
\psi_{1,0}(z)S(t) &= -\frac{(q^{k-2}z)^{-\frac{1}{k+2}}}{(q-q^{-1})t}\frac{(q^{-k+1}t/z;p)_\infty}{(q^{-k-1}t/z;p)_\infty}\Big(\frac{(1-q^{-k-1}t/z)}{(1-q^{-k+1}t/z)}:S_1(t)\psi_{1,0}(z): \\
&\qquad -:S_{-1}(t)\psi_{1,0}(z):\Big) \\
&= -\frac{(q^{k-2}z)^{-\frac{1}{k+2}}}{(q-q^{-1})t}\frac{(q^{-k+1}pt/z;p)_\infty}{(q^{-k-1}t/z;p)_\infty}\frac{t}{q^kz}\big(q(1-q^{k-1}z/t):S_{-1}(t)\psi_{1,0}(z): \\
&\qquad -q^{-1}(1-q^{k+1}z/t):S_1(t)\psi_{1,0}(z):\big), \quad |z|>q^{-k-1}|t|.
\end{aligned}$$

We can take the ratio of these two relations in order to extract extract the following commutation relation

$$\begin{aligned}
\mathcal{A}_{S\psi}(t/z) &= -\left(\frac{q^kz}{t}\right)^{\frac{1}{k+2}+1}\frac{\Theta_p(t/q^{k+1}z)}{\Theta_p(q^{-k+1}pt/z)} \\
&= q^{-1}\left(\frac{q^kz}{t}\right)^{\frac{1}{k+2}}\frac{\Theta_p(t/q^{k+1}z)}{\Theta_p(t/q^{k-1}z)}.
\end{aligned}$$

Again, this is a pseudo-constant with $\mathcal{A}_{S\psi}(z)=\mathcal{A}_{S\psi}(p^nz)$, $n\in\mathbb{Z}$.

Finally, we need to look at $\mathcal{A}_S$. Normal ordering relations give us

$$\begin{aligned}
& S(t_1)S(t_2) \\
={}& t_1^{\frac{2}{k+2}}\frac{(q^{-2}pt_2/t_1;p)_\infty}{(q^2t_2/t_1;p)_\infty} \\
& \times\big(q(1-t_2/t_1):S_1(t_1)S_1(t_2):+q^{-1}(1-t_2/t_1):S_{-1}(t_1)S_{-1}(t_2): \\
& -q(1-q^{-2}t_2/t_1):S_1(t_1)S_{-1}(t_2):-q^{-1}(1-q^2t_2/t_1):S_{-1}(t_1)S_1(t_2):\big).
\end{aligned}$$

If we then look at the ratio $\frac{S(t_1)S(t_2)}{S(t_2)S(t_1)}$, we find that

$$\begin{aligned}\mathcal{A}_S(t_1/t_2) &= -\frac{t_2}{t_1}\left(\frac{t_1}{t_2}\right)^{\frac{2}{k+2}}\frac{(q^{-2}pt_2/t_1;p)_\infty(q^2t_1/t_2;p)_\infty}{(q^2t_2/t_1;p)_\infty(q^{-2}pt_1/t_2;p)_\infty}\\ &= q^{-2}\left(\frac{t_1}{t_2}\right)^{\frac{2}{k+2}}\frac{\Theta_p(q^2t_1/t_2)}{\Theta_p(q^{-2}t_1/t_2)}.\end{aligned}$$

This is, again, a pseudo constant such that $\mathcal{A}_S(z)=\mathcal{A}_S(p^nz)$, $n\in\mathbb{Z}$.

The case of screened type II vertex operators throws out something slightly different as we are instead working with the modified screening current $\hat{S}(t) = S(t)\exp\{\tilde{a}_0\log(t)\}$. In this case, commutation with the usual screening current $S(t)$ looks like

$$\begin{aligned}S(t_1)\hat{S}(t_2) &= t_2^2q^{-2}\left(\frac{t_1}{t_2}\right)^{\frac{2}{k+2}}\frac{\Theta_p(q^2t_1/t_2)}{\Theta_p(q^{-2}t_1/t_2)}\hat{S}(t_2)S(t_1)\\ &= t_2^2\mathcal{A}_S\hat{S}(t_2)S(t_1).\end{aligned}$$

We label this modified factor by

$$\hat{\mathcal{A}}_S(t_1,t_2) = t_2^2\mathcal{A}_S(t_1/t_2), \tag{5.32}$$

which is *not* a pseudo-constant and means that the BRST relations for operators built with type II screened vertex operators will be more complicated than those with only type I screened vertex operators.

5.8.2 Pseudo-constants of Fixed Contour Integrals

The reason we call these quantities pseudo-constants is because, with some special treatment, we are able to pull them out of the integrand of certain contour integrals as though they were constants. This is useful when it comes to computing the graded operators arising in our eventual alternating sum of traces.

To see how this works, we look at the example of one of our BRST operators Q acting on a singly screened type I vertex operator,

$$\begin{aligned}
\phi^{(1)}_{2,2}(z) &= \phi_{2,2}(z)Q \\
&= \phi_{2,2}(z)\oint \frac{dt}{2\pi i t} S(t)\frac{[u-\frac{1}{2}+P_1]_{k+2}}{[u-\frac{1}{2}]_{k+2}}\frac{[1]_{k+2}}{[P_1]_{k+2}} \\
&= \phi_{2,2}(z)\sum_{m\geq 0} S(qp^m)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}-1}.
\end{aligned}$$

We have

$$\begin{aligned}
Q\,\phi^{(1)}_{2,2}(z) &= \oint_{\mathcal{C}_1}\frac{dt_1}{2\pi i t_1}S(t_1)\frac{[u_1-\frac{1}{2}+P_1]_{k+2}}{[u_1-\frac{1}{2}]_{k+2}}\phi_{2,2}(z)\oint_{\mathcal{C}_2}\frac{dt_2}{2\pi i t_2}S(t_2)\frac{[u_2-\frac{1}{2}+P_1]_{k+2}}{[u_2-\frac{1}{2}]_{k+2}} \\
&= \oint_{\mathcal{C}_1}\frac{dt_1}{2\pi i t_1}\mathcal{A}_{S\phi}(t_1/z)\phi_{2,2}(z)\oint_{\mathcal{C}_2}\frac{dt_2}{2\pi i t_2}\mathcal{A}_S(t_1/t_2)S(t_2) \\
&\quad\times\frac{[u_2-\frac{1}{2}+P_1]_{k+2}}{[u_2-\frac{1}{2}]_{k+2}}S(t_1)\frac{[u_1-\frac{1}{2}+P_1]_{k+2}}{[u_1-\frac{1}{2}]_{k+2}}.
\end{aligned}$$

The contours $\mathcal{C}_1$ and $\mathcal{C}_2$ are fixed and have poles at $t_1 = qp^m$ and $t_2 = qp^n$ $(m,n\in\mathbb{Z}_{\geq 0})$ lying inside as discussed in Section 5.3.3. We express the above in terms of formal sums and see that

$$\begin{aligned}
& Q\phi^{(1)}_{2,2}(z) \\
={}& \sum_{m\geq 0}\mathcal{A}_{S\phi}(qp^m/z)\phi_{2,2}(z)\sum_{n\geq 0}\mathcal{A}_S(p^{m-n})S(qp^n) \\
& \times\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}-1}S(qp^m)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}-1} \\
={}& \sum_{m\geq 0}\mathcal{A}_{S\phi}(q/z)\phi_{2,2}(z)\sum_{n\geq 0}\mathcal{A}_S(1)S(qp^n)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}-1}S(qp^m)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}-1} \\
={}& \mathcal{A}_{S\phi}(q/z)\mathcal{A}_S(1)\phi_{2,2}(z)\sum_{n\geq 0}S(qp^n)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}-1}\sum_{m\geq 0}S(qp^m)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}-1} \\
={}& \mathcal{A}_{S\phi}(q/z)\mathcal{A}_S(1)\phi_{2,2}(z)\oint_{\mathcal{C}_2}\frac{dt_2}{2\pi i t_2}S(t_2)\frac{[u_2-\frac{1}{2}+P_1]_{k+2}}{[u_2-\frac{1}{2}]_{k+2}} \\
& \times\oint_{\mathcal{C}_1}\frac{dt_1}{2\pi i t_1}S(t_1)\frac{[u_1-\frac{1}{2}+P_1]_{k+2}}{[u_1-\frac{1}{2}]_{k+2}} \\
={}& \mathcal{A}_{S\phi}(q/z)\mathcal{A}_S(1)\phi^{(1)}_{2,2}(z)\,Q.
\end{aligned}$$

In fact, we are able to write down the following relations for the BRST operator Q.

$$
\begin{aligned}
Q\phi_{2,2}(z) &= \mathcal{A}_{S\phi}(q/z)\phi_{2,2}(z)Q && (5.33)\\
Q\psi_{1,0}(\xi) &= \mathcal{A}_{S\psi}(q/\xi)\psi_{1,0}(\xi)Q && (5.34)\\
QS(t) &= \mathcal{A}_{S}(1)S(t)Q. && (5.35)
\end{aligned}
$$

We note here that we have to evaluate the pseudo-constants $\mathcal{A}$ at particular values within the BRST relations. In [62], the BRST operator is introduced as a Jackson integral and has commutation relations with the vertex operators and screening current that are also pseudo-constants. A complication in that case is that the pseudo-constants themselves have singularities at the points at which they must be evaluated within the BRST relations. This means that one must take care by introducing a regularisation parameter and carefully taking limits. Another advantage in the introduction of our novel BRST operator (5.19) is that no such singularities arise.

In the case of the type II screened vertex operators, we cannot pull out the commutation factors in the same way. To investigate this, we take one screened type II vertex operator and insert another type II vertex operator[6] and proceed in the

[6]We do this in order to preserve the argument of the BRST operator theta function ratio after pulling it through the operators.

same way as above, recalling relation (5.32). We have

$$
\begin{aligned}
& Q\,\psi_{1,0}(\xi_1)\psi_{1,0}^{(1)}(\xi_2) \\
= & \oint_{\mathcal{C}_1} \frac{dt_1}{2\pi i t_1} S(t_1) \frac{[u_1 - \frac{1}{2} + P_1]_{k+2}}{[u_1 - \frac{1}{2}]_{k+2}} \frac{[1]_{k+2}}{[P_1]_{k+2}}\, \psi_{1,0}(\xi_1)\psi_{1,0}(\xi_2) \\
& \times \oint_{\mathcal{C}_2} \frac{dt_2}{2\pi i t_2} \hat{S}(t_2) \frac{[u_2 - \frac{1}{2} + P_1]_{k+2}}{[u_2 - \frac{1}{2}]_{k+2}} \frac{[1]_{k+2}}{[P_1]_{k+2}} \\
= & \sum_{m\geq 0} S(qp^m) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1} \psi_{1,0}(\xi_1)\psi_{1,0}(\xi_2) \sum_{n\geq 0} \hat{S}(qp^n) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1} \\
= & \sum_{m\geq 0} \mathcal{A}_{S\psi}(qp^m/\xi_1)\mathcal{A}_{S\psi}(qp^m/\xi_2)\psi_{1,0}(\xi_1)\psi_{1,0}(\xi_2) \\
& \times \sum_{n\geq 0} \hat{\mathcal{A}}_S(qp^m, qp^n)\hat{S}(qp^n) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1} S(qp^m) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1} \\
= & \mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2)\mathcal{A}_S(1)\psi_{1,0}(\xi_1)\psi_{1,0}(\xi_2) \\
& \times \sum_{n\geq 0} q^2p^{2n}\hat{S}(qp^n) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1} \sum_{m\geq 0} S(qp^m) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1} \\
= & \mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2)\mathcal{A}_S(1)\psi_{1,0}(\xi_1)\psi_{1,0}(\xi_2) \\
& \times \sum_{n\geq 0} q^2p^{2n}\hat{S}(qp^n) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1}\, Q.
\end{aligned}
$$

We now recall the BRST relations for $V(\lambda_0)$, (5.29). Setting $\mathcal{O}^{[0]} = \psi_{1,0}(\xi_1)\psi_{1,0}^{(1)}(\xi_2)$ in these, we have

$$
\mathcal{O}^{[1]} = \mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2)\mathcal{A}_S(1)\psi_{1,0}(\xi_1)\psi_{1,0}(\xi_2) \sum_{n\geq 0} q^2p^{2n}\hat{S}(qp^n) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1}.
$$

Then acting with Q_3 results in

$$
\mathcal{O}^{[2]} = \mathcal{A}_{S\psi}(q/\xi_1)^4\mathcal{A}_{S\psi}(q/\xi_2)^4\mathcal{A}_S(1)^4\psi_{1,0}(\xi_1)\psi_{1,0}(\xi_2) \sum_{n\geq 0} q^8p^{8n}\hat{S}(qp^n) \frac{\Theta_p(q^2)}{(p;p)_\infty^3} q^{\frac{1}{k+2}-1}.
$$

In general, our graded operators in this case are defined by

$$
\begin{aligned}
\mathcal{O}^{2\alpha} &= \left(\mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2)\mathcal{A}_S(1)\right)^{4\alpha} S^{[2\alpha]}, \\
\mathcal{O}^{2\alpha+1} &= \left(\mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2)\mathcal{A}_S(1)\right)^{4\alpha+1} S^{[2\alpha+1]}, \qquad (5.36)
\end{aligned}
$$

where

$$\begin{aligned} S^{[2\alpha]} &= \sum_{n\geq 0} (qp^n)^{4\alpha} \hat{S}(qp^n) \\ S^{[2\alpha+1]} &= \sum_{n\geq 0} (qp^n)^{8\alpha+2} \hat{S}(qp^n). \end{aligned}$$

Here the nature of the operator really does change at each level, whereas when we have only type I vertex operators, we obtain a single repeated operator at each stage, albeit with a different pre-factor.

5.8.3 The S^+ XXZ Form-Factor Revisited

In this section, we will consider the BRST relations (5.29) for the combination of vertex operators and screening currents appearing in the two-particle S^+ form-factor (4.2). We focus on the first term of the S^+ form-factor and consider the string of vertex operators

$$\tilde{\Phi}^{\lambda_0;(2)}_{\lambda_2,1}(z_1)\tilde{\Phi}^{\lambda_2;(2)}_{\lambda_0,2}(z_2)\tilde{\Psi}^{\lambda_0;(1)}_{\lambda_1,0}(\xi_1)\tilde{\Psi}^{\lambda_1;(1)}_{\lambda_0,0}(\xi_2).$$

We suppress the normalisation factors arising through the relations (5.14) and (5.15) and introduce the appropriate number of screening charges, according to Section 5.3.3, to obtain the following operator of interest.

$$\begin{aligned} &Q^2\phi_{2,1}(z_1)\phi_{2,2}(z_2)Q\psi_{1,0}(\xi_1)\psi_{1,0}(\xi_2) \\ &= \oint \frac{dw}{2\pi i} J^-(w)\phi_{2,2}(z_1)Q^2\phi_{2,2}(z_2)\psi_{1,0}(\xi_1)\hat{Q}\psi_{1,0}(\xi_2) \\ &-q^2 \oint \frac{dw}{2\pi i}\phi_{2,2}Q^2 J^-(w)(z_1)\phi_{2,2}(z_2)\psi_{1,0}(\xi_1)\hat{Q}\psi_{1,0}(\xi_2). \end{aligned}$$

The current $J^-(w)$ commmutes with the BRST operator Q and so will not have an effect on the relations. We will consider the BRST relations (5.29) for $\mathcal{O}$, where we let

$$\mathcal{O} = J^-(w)\phi_{2,2}(z_1)Q^2\phi_{2,2}(z_2)\psi_{1,0}(\xi_1)\hat{Q}\psi_{1,0}(\xi_2).$$

Using relations (5.33), (5.34) and (5.35), we have

$$\begin{aligned}
Q\mathcal{O} &= Q\, J^-(w)\phi_{2,2}(z_1)\phi_{2,2}(z_2)Q^2\psi_{1,0}(\xi_1)\hat{Q}\psi_{1,0}(\xi_2) \\
&= \mathcal{A}_S(1)^3\mathcal{A}_{S\phi}(q/z_1)\mathcal{A}_{S\phi}(q/z_2)\mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2) \\
&\quad \times J^-(w)\phi_{2,2}(z_1)\phi_{2,2}(z_2)Q^2\psi_{1,0}(\xi_1)\sum_{n\geq 0} q^2p^{2n}\hat{S}(qp^n)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}}\psi_{1,0}(\xi_2)\,Q,
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{O}^{[1]} &= \mathcal{A}_S(1)^3\mathcal{A}_{S\phi}(q/z_1)\mathcal{A}_{S\phi}(q/z_2)\mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2) \\
&\quad \times J^-(w)\phi_{2,2}(z_1)\phi_{2,2}(z_2)Q^2\psi_{1,0}(\xi_1)\sum_{n\geq 0} q^2p^{2n}\hat{S}(qp^n)\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}}\psi_{1,0}(\xi_2).
\end{aligned}$$

In a similar way to (5.36), we obtain the following definition for the graded operators in this case.

$$\begin{aligned}
\mathcal{O}^{\square} &= \left(\mathcal{A}_{S\phi}(q/z_1)\mathcal{A}_{S\phi}(q/z_2)\mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2)\mathcal{A}_S(1)^3\right)^{4\alpha} \\
&\quad \times J^-(w)\phi_{2,2}(z_1)\phi_{2,2}(z_2)Q^2\psi_{1,0}(\xi_1)S^{[2\alpha]}\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}}\psi_{1,0}(\xi_2), \quad (5.37) \\
\mathcal{O}^{[2\alpha+1]} &= \left(\mathcal{A}_{S\phi}(q/z_1)\mathcal{A}_{S\phi}(q/z_2)\mathcal{A}_{S\psi}(q/\xi_1)\mathcal{A}_{S\psi}(q/\xi_2)\mathcal{A}_S^3(1)\right)^{4\alpha+1} \\
&\quad \times J^-(w)\phi_{2,2}(z_1)\phi_{2,2}(z_2)Q^2\psi_{1,0}(\xi_1)S^{[2\alpha+1]}\frac{\Theta_p(q^2)}{(p;p)^3_\infty}q^{\frac{1}{k+2}}\psi_{1,0}(\xi_2),
\end{aligned}$$

where again

$$\begin{aligned}
S^{[2\alpha]} &= \sum_{n\geq 0}(qp^n)^{4\alpha}\hat{S}(qp^n) \\
S^{[2\alpha+1]} &= \sum_{n\geq 0}(qp^n)^{8\alpha+2}\hat{S}(qp^n).
\end{aligned}$$

And so the trace expression for this particular term of the S^+ form-factor (up to normalisation), is given by

$$\begin{aligned}
&\mathrm{Tr}_{V(2\Lambda_0)}\left(q^{-2\rho}\phi_{2,2}(z_1)Q^2\phi_{2,2}(z_2)\psi_{1,0}(\xi_1)\hat{Q}\psi_{1,0}(\xi_2)\right) \quad (5.38) \\
&= \sum_{s\in\mathbb{Z}}\mathrm{Tr}_{F_0^{[s]}}\left(q^{-2\rho}\oint\frac{dw_0}{2\pi i w_0}\,\xi(w_0)\oint\frac{dw}{2\pi i}\,\eta(w)\mathcal{O}^{[s]}\right)\Bigg|_{\tilde{b}_0+\tilde{c}_0=0}, \quad (5.39)
\end{aligned}$$

where graded operators $\mathcal{O}^{[s]}$ are defined by (5.37) and (5.38). Similar relations can be obtained for the other choices of ground state boundary condition using (5.30) and (5.31). These are equally complicated due to the appearance of the screened type II vertex operator.

The framework set out in this chapter is enough to generalise and obtain the corresponding relations for arbitrary level and spin. Although the number of integrals from the lower type I vertex operator components will increase as the spin (and therefore number of components) increases, the complexity in terms of the bosonisation scheme stays at the same level, one of the distinct advantages of this approach. However, extracting a useful expression from the resultant alternating infinite sum seems, at present, to be an insurmountable task.

5.9 A Note on Correlation Functions and Matrix Elements

Whilst the BRST relations for the type II vertex operators make the computation of form-factors somewhat complicated, computation of correlation functions is more straight forward as we only encounter type I vertex operators, which have pure pseudo-constant commutation relations with the BRST operator. In this case, we will only have a single repeated operator appearing in the trace expressions analogous to (5.38), with a different number of pseudo-constant pre-factors in each term of the alternating sum.

Computing matrix elements is also a straightforward task as we do not have to take a trace and avoid the occurrence of alternating sums of graded operators altogether. Such calculations are of interest in the context of [107], where similar objects are computed for perfect vertex operators. The q-Wakimoto scheme outlined above allows one to compute highest-weight to highest-weight matrix elements for mixtures of both perfect and imperfect vertex operators, for arbitrary spin and level.

Chapter 6

Conclusion and Outlook

The work comprising this thesis began as an investigation into the feasibility of computing explicit two-particle contributions to the form-factors of the massive spin-1 Heisenberg chain (3.1), motivated by their relation to the experimentally realisable dynamical structure factors of the same model. By using a well documented free field realisation in terms of one boson and one fermion [59–61], the bosonic contribution to the trace proved early on to be a straight forward object to compute through the use of normal ordering relations and the bosonic trace formula (3.35).

The real difficulty in the use of this bosonisation scheme arose through the presence of fermion emission operators within the imperfect type II vertex operators used to construct the spinon states. An attempt to overcome this using an approach based on methods considered in [61] and [22] does succeed in giving an integral expression[1], but it is an unpleasant one in terms of infinite sums of Pfaffians. At present, we have little hope of using this expression in conjunction with existing algebraic Bethe ansatz results [68, 69] to compute dynamical structure factors.

The approach introduced in Chapter 3 to tackle this problem is a novel one, based on what we have called the Shiraishi realisation for fermions [78]. Using this together with the one boson, one fermion scheme, we have been able to introduce the

[1]This expression will be given in detail in [79] and is primarily the work of R. A. Weston

entire formalism required to obtain integral expressions for arbitrary $2m$-particle form-factors for spin-1. Previous progress in this area was halted at the level of trace expressions, apart from in some specialised cases which were treated in an ad-hoc manner [66]. Another nice result from investigating this construction was to verify the previously conjectured form of the two-point function of Ising vertex operators from [22].

Using the results from Chapter 4, we have been able to compute explicit single integral expressions for two-spinon contributions to the S^+ form-factor, as the original project set out to achieve. For the other spin matrices, more integrals appear through the Drinfeld currents and the computations are more involved both for the bosonic and fermionic parts. Nonetheless, the necessary information to extract integral expressions for their form-factors is given in Chapter 4.

Following the consideration of the spin-1 case, we have made a less successful attempt to extend our investigations to free field realisations for arbitrary level and spin. A modification of a known q-Wakimoto bosonisation scheme [62–64] has been introduced in an attempt to produce simple BRST relations for the products of vertex operators appearing in the form-factor trace expressions we would like to compute. Certain properties of the commutation relations between the vertex operators and our BRST operator do simplify the situation, but further work is needed on the explicit calculation of the alternating sums of the trace expressions obtained. In higher spin cases the q-Wakimoto scheme is necessary, but for the spin-1 case there certainly seems to be no advantage in considering this instead of what we will call our boson-fermion-Shiraishi realisation.

The hope, moving forwards, is that the spin-1 form-factor results obtained in this work will be used in conjunction with work resulting from the algebraic Bethe ansatz approach to higher spin Heisenberg chains [68, 69, 108] to produce results for the spin-1 dynamical structure factors. The ultimate aim is to then make direct a comparison with experimental results, which seems to be a realistic goal.

Appendix A

Useful Notation, Formulae and Miscellaneous

q-calculus

$$
\begin{aligned}
[x] &= \frac{q^x - q^{-x}}{q - q^{-1}}, \\
[n]! &= [n][n-1]\dots[2][1], \\
\left[\begin{array}{c} n \\ k \end{array} \right] &= \frac{[n]!}{[k]![n-k]!}.
\end{aligned}
$$

The q-infinite product and useful relations

$$
\begin{aligned}
(x;q)_\infty &= \prod_{n>0}(1 - xq^n), \\
(x_1, x_2, \dots, x_k; q)_\infty &= (x_1;q)_\infty (x_2;q)_\infty \dots (x_k;q)_\infty \\
(x; q_1, q_2, \dots q_k)_\infty &= \prod_{i_1, i_2, \dots i_k = 0}^{\infty} (1 - x q_1^{i_1} q_2^{i_2} \dots q_l^{i_k}), \\
(x; q, p)_{n,m} &= \prod_{j=0}^{n} \prod_{k=0}^{m} (1 - x q^j p^k) \\
(-x;q)_\infty (x;q)_\infty &= (x^2; q^2)_\infty. \\
(x; q, -q)_\infty &= (x; -q)_\infty (qx; q, -q)_\infty. \\
(x; q, q)_\infty (-x; q, q)_\infty &= (x^2; q^2, q^2)_\infty = (x; q, -q)_\infty (-x; q, -q)_\infty.
\end{aligned}
$$

Basic hypergeometric series

$$
\begin{aligned}
{}_r\phi_s(a_1,a_2,\dots,a_r;b_1,\dots,b_2;q,z) &\equiv {}_r\phi_s\left[\begin{array}{c} a_1,a_2,\dots,a_r \\ b_1,\dots,b_s \end{array};q,z\right] \\
&= \sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n.
\end{aligned}
$$

Theta functions

$$
\begin{aligned}
\Theta_p(x) &= (x;p)_\infty(p/x;p)_\infty(p;p)_\infty \\
[u]_x &= \vartheta_1\left(\frac{u}{x}|\tau_x\right) \\
\vartheta_1\left(\frac{u}{x}|\tau_x\right) &= (-i\tau_x)^{-\frac{1}{2}}q^{\frac{x}{4}}[u]_x \\
&= (-i\tau_x)^{-\frac{1}{2}}q^{\frac{x}{4}}q^{u^2/x-u}\Theta_{q^{2x}}(q^{2u}) \\
\Theta_p(x) &= -x\Theta_p(1/x) \\
\Theta_{p^2}(px) &= \sum_{n\in\mathbb{Z}}(-1)^n p^{n^2}x^n
\end{aligned}
$$

The q-Gamma function

$$
\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}
$$

The q-binomial theorem

$$
{}_1\phi_0(a;q,z) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty},\ \ |z|<1,\ \ |q|<1.
$$

Heine's transformation formula

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(b,az,q)_\infty}{(c,z;q)_\infty}{}_2\phi_1(c/b,z;az;q,b),
$$

Hopf algebra axiomatic properties

$$
\begin{aligned}
&\Delta(xy) = \Delta(x)\Delta(y), \quad \epsilon(xy) = \epsilon(x)\epsilon(y), \quad a(xy) = a(y)a(x) \\
&(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \\
&(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta \\
&m \circ (a \otimes id) \circ \Delta = \epsilon = m \circ (id \otimes a) \circ \Delta, \quad m(x \otimes y) = xy.
\end{aligned}
$$

Appendix B

The Bosonic Trace Formula

We will outline the derivation of the useful formula

$$\begin{aligned} & \mathrm{Tr}_{F^a}\left(x^{-\rho}\exp\left(\sum_{n=1}^{\infty}A_n a_{-n}\right)\exp\left(\sum_{n=1}^{\infty}B_n a_n\right)\right) \\ = \; & \exp\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}x^{2mn}A_nB_n\frac{[2n]^2}{n}\right)\frac{1}{(x^2;x^2)_\infty}. \end{aligned}$$

We start by considering the above trace over F^a as the product over n of the traces over Fock spaces spanned by just one 'species' of boson. That is to say, we consider

$$\begin{aligned} & \mathrm{Tr}_{F^a}\left(x^{-2d^a}\exp\left(\sum_{n=1}^{\infty}A_n a_{-n}\right)\exp\left(\sum_{n=1}^{\infty}B_n a_n\right)\right) \\ = \; & \prod_{n>0} Tr_{F^a_{(n)}}\left(x^{-2d^a}\exp(A_n a_{-n})\exp(B_n a_n)\right), \end{aligned}$$

where

$$F^a_{(n)} = \mathrm{span}\{a_{-n}\left|0\right\rangle, \left|0\right\rangle\},$$

and $2d^a$ is the bosonic part of the grading operator. Then

$$
\begin{aligned}
&\prod_{n>0} Tr_{F^a_{(n)}}\left(x^{-2d^a}\exp(A_n a_{-n})\exp(B_n a_n)\right)\\
=&\prod_{n=1}^{\infty}\sum_{k=0}^{\infty} x^{2nk}\frac{\langle 0|\, a_n^k e^{A_n a_{-n}} e^{B_n a_n} a_{-n}^k\, |0\rangle}{\langle 0|\, a_n^k a_{-n}^k\, |0\rangle}\\
=&\prod_{n=1}^{\infty}\sum_{k=0}^{\infty} x^{2nk}\frac{\langle 0|\,(a_n + A_n[a_n,a_{-n}])^k\,(a_{-n}+B_n[a_n,a_{-n}])^k\,|0\rangle}{\langle 0|\, a_n^k a_{-n}^k\, |0\rangle},
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
e^{B_n a_n} a_{-n}^k\,|0\rangle &= (a_{-n}+B_n[a_n,a_{-n}])^k\,|0\rangle\,,\\
\langle 0|\, a_n^k e^{A_n a_{-n}} &= \langle 0|\,(a_n + A_n[a_n,a_{-n}])^k,
\end{aligned}
$$

both of which are derived by using the fact that for operators A, B such that $[A,B]$ is scalar (i.e commutes with everything), we have

$$
e^A B e^{-A} = B + [A,B].
$$

For the next step, we also need the relation

$$
\begin{aligned}
&\langle 0|\, a_n^k a_{-n}^k\, |0\rangle\\
=&\ \langle 0|\, a_n^{k-1}([a_n,a_{-n}] + a_{-n}a_n)a_{-n}^{k-1}\,|0\rangle\\
=&\ [a_n,a_{-n}]\,\langle 0|\, a_n^{k-1}a_{-n}^{k-1}\,|0\rangle + \langle 0|\, a_n^{k-1}a_{-n}([a_n,a_{-n}]+a_{-n}a_n)^{k-2}\,|0\rangle\\
=&\ [a_n,a_{-n}]\,\langle 0|\, a_n^{k-1}a_{-n}^{k-1}\,|0\rangle\\
&+[a_n,a_{-n}]\,\langle 0|\, a_n^{k-1}a_{-n}^{k-1}\,|0\rangle + \langle 0|\, a_n^{k-1}a_{-n}([a_n,a_{-n}]+a_{-n}a_n)^{k-2}\,|0\rangle\\
=&\ \ldots\\
=&\ k[a_n,a_{-n}]\,\langle 0|\, a_n^{k-1}a_{-n}^{k-1}\,|0\rangle\\
=&\ k(k-1)[a_n,a_{-n}]^2\,\langle 0|\, a_n^{k-2}a_{-n}^{k-2}\,|0\rangle\\
=&\ \ldots\\
=&\ k![a_n,a_{-n}]^k\,\langle 0|0\rangle
\end{aligned}
$$

We now write $C_n = [a_n, a_{-n}]$ and expand the brackets using the binomial theorem to obtain

$$\begin{aligned}
&\prod_{n=1}^{\infty}\sum_{k=0}^{\infty} x^{2nk}\frac{\langle 0|\left(a_n + A_n[a_n,a_{-n}]\right)^k\left(a_{-n}+B_n[a_n,a_{-n}]\right)^k|0\rangle}{\langle 0|\, a_n^k a_{-n}^k\,|0\rangle}\\
&= \prod_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} x^{2nk}\frac{\langle 0|\, a_n^{k-l}a_{-n}^{k-l}\,|0\rangle \binom{k}{l}^2 (A_nB_nC_n^2)^l}{\langle 0|\, a_n^k a_{-n}^k\,|0\rangle}\\
&= \prod_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} x^{2nk}\frac{\langle 0|\,|0\rangle\ (k-l)!\, C_n^{k-l}\binom{k}{l}^2 (A_nB_nC_n^2)^l}{\langle 0|\,|0\rangle\ k!\, C_n^k}\\
&= \prod_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} x^{2nk}\frac{(k-l)!\,(k!)^2\,(A_nB_nC_n^2)^l}{C_n^l\,(l!)^2\ \left((k-l)!\right)^2}\\
&= \prod_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} x^{2nk}\frac{k!\,(A_nB_nC_n)^l}{(l!)^2\,(k-l)!}\\
&= \prod_{n=0}^{\infty}\exp\left(\frac{x^{2n}A_nB_nC_n}{1-x^{2n}}\right)\frac{1}{1-x^{2n}}\\
&= \exp\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x^{2mn}A_nB_nC_n\right)\frac{1}{(x^2;x^2)_\infty},
\end{aligned}$$

where we use the relation

$$\sum_{k=0}^{\infty} r^k \sum_{l=0}^{k}\frac{k!\,s^l}{(l!)^2\,(k-l)!} = \exp\left(\frac{rs}{1-r}\right)\frac{1}{1-r},$$

from [23].

Appendix C

Normal Ordering Relations and Trace Contributions for One Boson, One Fermion Formalism

C.1 Normal Ordering

This appendix lists all operator product expansion formulae required in the one boson, one fermion formalism. The simple components of our vertex operators have free field realisation

$$\begin{aligned}\Phi_2(z) &= A_<(z)\, A_>(z)\, e^{\alpha}\, (-q^4 z)^{\partial/2}, \\ \Psi_0(z) &= B_{II,<}(z)\, B_{II,>}(z)\, \Omega(q^{-2}z)\, e^{-\alpha/2}\, (-q^2 z)^{-\partial/4},\end{aligned}$$

where

$$\begin{aligned}
E^{\pm}_{<}(z) &= \exp\left(\pm\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{\mp n}z^{n}\right), \\
E^{\pm}_{>}(z) &= \exp\left(\mp\sum_{n=1}^{\infty}\frac{a_{n}}{[2n]}q^{\mp n}z^{-n}\right), \\
A_{<}(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{5n}z^{n}\right), \\
A_{>}(z) &= \exp\left(-\sum_{n=1}^{\infty}\frac{a_{n}}{[2n]}q^{-3n}z^{-n}\right), \\
B_{II,<}(z) &= \exp\left(-\sum_{n=1}^{\infty}\frac{[n]a_{-n}}{[2n]^2}(qz)^{n}\right), \\
B_{II,>}(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{[n]a_{n}}{[2n]^2}(q^3z)^{-n}\right).
\end{aligned}$$

We also have currents

$$x^{\pm}(w) = E^{\pm}_{<}(w)\, E^{\pm}_{>}(w)\, \phi(w)e^{\pm\alpha}w^{(1\pm\partial)/2}.$$

The normal ordering relations for the different fields are given by

$$\begin{aligned}
E^{-}_{>}(w)B_{II,<}(\xi) &= \frac{(q^3\xi/w;q^4)_\infty}{(q^5\xi/w;q^4)_\infty} : B_{II,<}(\xi)E^{-}_{>}(w) :, \\
E^{-}_{>}(w)A_{<}(z) &= \frac{1}{(1-q^6z/w)} : A<(z)E>^{-}(w) :, \\
A_{>}(z)E^{-}_{<}(w) &= \frac{1}{(1-q^{-2}w/z)} : E^{-}_{<}(w)A_{>}(z) :, \\
A_{>}(z)E^{+}_{<}(u) &= (1-u/q^4z) : A_{>}(z)E^{+}_{<}(u) : \\
A_{>}(z_1)A_{<}(z_2) &= \big(1-q^2z_2/z_1\big) : A_{<}(z_2)A_{>}(z_1) :, \\
A_{>}(z)B_{II,<}(\xi) &= \frac{(q\xi/z;q^4)_\infty}{(\xi/qz;q^4)_\infty} : B_{II,<}(\xi)A_{>}(z) :, \\
B_{II,>}(\xi_1)B_{<,II}(\xi_2) &= \frac{(q^4\xi_2/\xi_1;q^4,q^4)_\infty\,(\xi_2/\xi_1;q^4,q^4)_\infty}{(q^2\xi_2/\xi_1;q^4,q^4)_\infty^2} : B_{II,<}(\xi_2)B_{II,>}(\xi_1) :, \\
E^{+}_{>}(w)B_{II,<}(\xi) &= \frac{(q^3\xi/w;q^4)_\infty}{(q\xi/w;q^4)_\infty} : B_{II,<}(\xi)E^{+}_{>}(w) :, \\
B_{II,>}(\xi)E^{+}_{<}(w) &= \frac{(w/q\xi;q^4)_\infty}{(w/q^3\xi;q^4)_\infty} : E^{+}_{<}(w)B_{II,>}(\xi) : .
\end{aligned}$$

We have the following bosonic normal ordering relations between simple vertex operator components and currents:

$$\begin{aligned}
x^-(w_1)x^-(w_2) &= w_1(1-q^2w_2/w_1) : x^-(w_1)x^-(w_2) : \\
x^+(w_1)x^+(w_2) &= w_1(1-q^{-2}w_2/w_1) : x^+(w_1)x^+(w_2) : \\
x^-(w_1)x^+(w_2) &= \frac{1}{w_1(1-w_2/w_1)} : x^-(w_1)x^+(w_2) : \\
x^-(w)\Phi_2(z) &= \frac{1}{w(1-q^6z/w)} : x^-(w)\Phi_2(z) : \\
\Phi_2(z)x^-(w) &= \frac{1}{(1-w/q^2z)(-q^4z)} : \Phi_2(z)x^-(w) : \\
\Phi_2(z_1)\Phi_2(z_2) &= \left(1-\frac{q^2z_2}{z_1}\right)\left(-q^4z_1\right) : \Phi_2(z_1)\Phi_2(z_2) : \\
\Phi_2(z)x^+(w) &= -q^4z(1-w/q^4z) : \Phi_2(z)x^+(w) :
\end{aligned}$$

$$\begin{aligned}
\Psi_0(\xi_1)\Psi_0(\xi_2) &= \frac{(q^4\xi_2/\xi_1;q^4,q^4)_\infty\,(\xi_2/\xi_1;q^4,q^4)_\infty}{(q^2\xi_2/\xi_1;q^4,q^4)_\infty^2}(-q^2\xi_1)^{1/4} \\
x^+(w)\Psi_0(\xi) &= \frac{(q^3\xi/w;q^4)_\infty}{(q\xi/w;q^4)_\infty}w^{-\frac{1}{2}} : \hat{x}^+(w)\hat{\Psi}_0(\xi) : \phi(w)\Omega(q^{-2}\xi) \\
\Psi_0(\xi)x^+(w) &= \frac{(w/q\xi;q^4)_\infty}{(w/q^3\xi;q^4)_\infty}(-q^2\xi)^{-\frac{1}{2}}. \\
x^-(w)\Psi_0(\xi) &= \frac{(q^3\xi/w;q^4)_\infty}{(q^5\xi/w;q^4)_\infty}w^{1/2} : \hat{x}^-(w)\hat{\Psi}_0(\xi) : \phi(w) \\
\Psi_0(\xi)x^-(w) &= (-q^2\xi)^{\frac{1}{2}}\frac{(w/q\xi;q^4)_\infty}{(qw/\xi;q^4)_\infty} : \hat{\Psi}_0(\xi)\hat{x}^-(w) : \phi(w) \\
\Phi_2(z)\Psi_0(\xi) &= (-q^4z)^{-\frac{1}{2}}\frac{(q\xi/z;q^4)_\infty}{(\xi/qz;q^4)_\infty} : \Phi_2(z)\Psi_0(\xi) : \\
\Psi_0(\xi)\Phi_2(z) &= (-q^2\xi)^{-\frac{1}{2}}\frac{(q^5z/\xi;q^4)_\infty}{(q^3z/\xi;q^4)_\infty}. : \Psi_0(\xi)\Phi_2(z) :
\end{aligned}$$

We define normal ordered operators comprised of simple vertex operator components and currents as they appear in Chapter 3:

$$\begin{aligned}
\mathcal{O}_C(z_c) &= \Phi_2(z_c), \\
\mathcal{O}_B(z_b, w_b) &= :\Phi_2(z_b)\hat{x}^-(w_b):, \\
\mathcal{O}_A(z_a, v_a, w_a) &= :\Phi_2(z_a)\hat{x}^-(w_a)\hat{x}^-(v_a):, \\
\mathcal{O}_D(\xi_d) &= \hat{\Psi}_0(z_d), \\
\mathcal{O}_E(\xi_e, u_e) &= :\hat{\Psi}_0(z_e)\hat{x}^+(u_e):.
\end{aligned}$$

We denote the normal ordering factor for each pair of operators by $\mathcal{N}_{X_1X_2}$, so that

$$\begin{aligned}
\mathcal{O}_{X_1}(z_1, \dots z_n)\mathcal{O}_{X_2}(w_1, \dots w_m) &= \mathcal{N}_{X_1X_2}(z_1, \dots, z_n, w_1, \dots w_m) \\
&\quad \times :\mathcal{O}_{X_1}(z_1, \dots z_n)\mathcal{O}_{X_2}(w_1, \dots w_m):.
\end{aligned}$$

Explicitly, we have

$$\begin{aligned}
\mathcal{N}_{AA}(z_1, v_1, w_1; z_2, v_2, w_2) &= -\frac{v_1 w_1(1-q^2z_2/z_1)(1-q^2v_2/v_1)}{q^4z_1(1-v_2/q^2z_1)(1-w_2/q^2z_1)} \\
&\quad \times \frac{(1-q^2w_2/v_1)(1-q^2v_2/w_1)(1-q^2w_2/w_1)}{(1-q^6z_2/v_1)(1-q^6z_2/w_1)} \\
\mathcal{N}_{BB}(z_1, w_1; z_2, w_2) &= \frac{(1-q^2w_2/w_1)}{(1-q^6z_2/w_1)(1-w_2/q^2z_1)} \\
\mathcal{N}_{CC}(z_1; z_2) &= \left(1-q^2z_2/z_1\right)\left(-q^4z_1\right) \\
\mathcal{N}_{AB}(z_1, v_1, w_1; z_2, w_2) &= \frac{(1-q^2z_2/z_1)(1-q^2w_2/v_1)(1-q^2w_2/w_1)}{(1-w_2/q^2z_1)(1-q^6z_2/w_1)(1-q^6z_2/v_1)} \\
\mathcal{N}_{BA}(z_2, w_2; z_1, v_1, w_1) &= -\frac{w_1(1-q^2z_2/z_1)(1-q^2v_2/w_1)(1-q^2w_2/w_1)}{q^4z_1(1-q^6z_2/w_1)(1-v_2/q^2z_1)(1-w_2/q^2z_1)} \\
\mathcal{N}_{AC}(z_1, v_1, w_1; z_2) &= -\frac{q^4z_1(1-q^2z_2/z_1)}{v_1w_1(1-q^6z_2/w_1)(1-q^6z_2/v_1)} \\
\mathcal{N}_{CA}(z_2; z_1, v_1, w_1) &= -\frac{(1-q^2z_2/z_1)}{q^4z_1(1-w_2/q^2z_1)(1-v_2/q^2z_1)} \\
\mathcal{N}_{BC}(z_1, w_1; z_2) &= -\frac{q^4z_1(1-q^2z_2/z_1)}{w_1(1-q^6z_2/w_1)} \\
\mathcal{N}_{CB}(z_1; z_2, w_2) &= \frac{(1-q^2z_2/z_1)}{(1-w_2/q^2z_1)}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{N}_{DD}(\xi_1;\xi_2) &= \frac{(q^4\xi_2/\xi_1;q^4,q^4)_\infty\,(\xi_2/\xi_1;q^4,q^4)_\infty}{(q^2\xi_2/\xi_1;q^4,q^4)_\infty^2}(-q^2\xi_1)^{1/4}\\
\mathcal{N}_{EE}(\xi_1,u_1;\xi_2,u_2) &= \frac{(q^4\xi_2/\xi_1;q^4,q^4)_\infty\,(\xi_2/\xi_1;q^4,q^4)_\infty}{(q^2\xi_2/\xi_1;q^4,q^4)_\infty^2}(-q^2\xi_1)^{-\frac{1}{4}}\\
&\quad\times u_1^{\frac{1}{2}}(1-q^{-2}u_2/u_1)\frac{(q^3\xi_2/u_1;q^4)_\infty}{(q\xi_2/u_1;q^4)_\infty}\frac{(u_2/q\xi_1;q^4)_\infty}{(u_2/q^3\xi_1;q^4)_\infty}\\
\mathcal{N}_{DE}(z_1;z_2,u_2) &= \frac{(q^4\xi_2/\xi_1;q^4,q^4)_\infty\,(\xi_2/\xi_1;q^4,q^4)_\infty}{(q^2\xi_2/\xi_1;q^4,q^4)_\infty^2}\\
&\quad\times(-q^2\xi_1)^{-\frac{1}{4}}\frac{(u_2/q\xi_1;q^4)_\infty}{(u_2/q^3\xi_1;q^4)_\infty}\\
\mathcal{N}_{ED}(z_1,u_1;z_2) &= \frac{(q^4\xi_2/\xi_1;q^4,q^4)_\infty\,(\xi_2/\xi_1;q^4,q^4)_\infty}{(q^2\xi_2/\xi_1;q^4,q^4)_\infty^2}\\
&\quad\times(-q^2\xi_1)^{1/4}\frac{(q^3\xi_2/u_1;q^4)_\infty}{(q\xi_2/u_1;q^4)_\infty}u_1^{-\frac{1}{2}}\\
\mathcal{N}_{AD}(z_1,v_1,w_1;\xi_1) &= \frac{(q\xi_1/z_1;q^4)_\infty}{(\xi_1/qz_1;q^4)_\infty}(-q^4z_1)^{-1/2}\\
&\quad\times\frac{(q^3\xi_1/v_1;q^4)_\infty}{(q^5\xi_1/v_1;q^4)_\infty}v_1^{1/2}\frac{(q^3\xi_1/w_1;q^4)_\infty}{(q^5\xi_1/w_1;q^4)_\infty}w_1^{1/2}\\
\mathcal{N}_{BD}(z_1,w_1;\xi_1) &= \frac{(q\xi_1/z_1;q^4)_\infty}{(\xi_1/qz_1;q^4)_\infty}(-q^4z_1)^{-1/2}w_1^{1/2}\frac{(q^3\xi_1/w_1;q^4)_\infty}{(q^5\xi_1/w_1;q^4)_\infty}\\
\mathcal{N}_{CD}(z_1;\xi_1) &= \frac{(q\xi_1/z_1;q^4)_\infty}{(\xi_1/qz_1;q^4)_\infty}(-q^4z_1)^{-1/2}\\
\mathcal{N}_{AE}(z_1,v_1,w_1;\xi_1,u_1) &= \frac{(q\xi_1/z_1;q^4)_\infty}{(\xi_1/qz_1;q^4)_\infty}(-q^4z_1)^{-1/2}\\
&\quad\times\frac{(q^3\xi_1/v_1;q^4)_\infty}{(q^5\xi_1/v_1;q^4)_\infty}v_1^{1/2}\frac{(q^3\xi_1/w_1;q^4)_\infty}{(q^5\xi_1/w_1;q^4)_\infty}w_1^{1/2}\\
&\quad\times\frac{(-q^4z_1)(1-u_1/q^4z_1)}{v_1w_1(1-u_1/v_1)(1-u_1/w_1)}\\
\mathcal{N}_{BE}(z_1,w_1;\xi_1,u_1) &= -\frac{(q\xi_1/z_1;q^4)_\infty}{(\xi_1/qz_1;q^4)_\infty}(-q^4z_1)^{-1/2}w_1^{1/2}\\
&\quad\times\frac{(q^3\xi_1/w_1;q^4)_\infty}{(q^5\xi/w_1;q^4)_\infty}q^4z_1(1-u_1/q^4z)\frac{1}{w_1(1-u_1/w_1)}\\
\mathcal{N}_{CE}(z_1;\xi_1,u_1) &= -\frac{(q\xi_1/z_1;q^4)_\infty}{(\xi_1/qz_1;q^4)_\infty}(-q^4z_1)^{1/2}(1-u_1/q^4z_1)
\end{aligned}
$$

C.2 Trace Contributions

This section lists self contributions $g_{\mathcal{O}_i}$ (3.45) and pairwise contributions $g_{\mathcal{O}_i\mathcal{O}_j}(z_i, z_j)$ (3.46) to a bosonic trace of the form

$$\mathrm{Tr}_{\mathcal{F}^a}\left(x^{-2d^a} : \mathcal{O}_1 \dots \mathcal{O}_m :\right),$$

where the string of operators is built from the simple vertex operator components ϕ_2, Ψ_0 and currents $x^{\pm}$.

$$\begin{aligned}
g_{\Phi_2} &= (x^2q; x^2)_\infty \\
g_{x^-} &= (x^2q; x^2)_\infty \\
g_{x^+} &= (x^2q^{-2}; x^2)_\infty \\
g_{\Phi_2\Phi_2}(z_1, z_2) &= (x^2q^2z_1/z_2; x^2)_\infty(x^2q^2z_2/z_1; x^2)_\infty \\
g_{x^-x^-}(v, w) &= (x^2q^2v/w; x^2)_\infty(x^2q^2w/v; x^2)_\infty \\
g_{x^+x^+}(v, w) &= (x^2q^{-2}v/w; x^2)_\infty(x^2q^{-2}w/v; x^2)_\infty \\
g_{\Phi_2x^-}(z, w) &= \frac{1}{(x^2q^6z/w; x^2)_\infty(x^2w/q^2z; x^2)_\infty} \\
g_{\Phi_2x^+}(z, w) &= (x^2q^4z/w; x^2)_\infty(x^2w/q^4z; x^2)_\infty \\
g_{\Psi_0\Psi_0}(\xi_1, \xi_2) &= \frac{(x^2q^4\xi_2/\xi_1; q^4, q^4, x^2)_\infty\,(x^2\xi_2/\xi_1; q^4, q^4, x^2)_\infty}{(x^2q^2\xi_2/\xi_1; q^4, q^4, x^2)^2_\infty} \\
&\quad\times\frac{(x^2q^4\xi_1/\xi_2; q^4, q^4, x^2)_\infty\,(x^2\xi_1/\xi_2; q^4, q^4, x^2)_\infty}{(x^2q^2\xi_1/\xi_2; q^4, q^4, x^2)^2_\infty} \\
g_{\Psi_0\Phi_2}(\xi, z) &= \frac{(x^2q^5z/\xi; q^4, x^2)_\infty(x^2q\xi/z; q^4, x^2)_\infty}{(x^2q^3z/\xi; q^4, x^2)_\infty(x^2\xi/qz; q^4, x^2)_\infty} \\
g_{\Psi_0x^-}(\xi, w) &= \frac{(x^2w/q\xi; q^4, x^2)_\infty(x^2q^3\xi/w; q^4, x^2)_\infty}{(x^2qw/\xi; q^4, x^2)_\infty(x^2q^5\xi/w; q^4, x^2)_\infty} \\
g_{\Psi_0x^+}(\xi, w) &= \frac{(x^2w/q\xi; q^4, x^2)_\infty(x^2q^3\xi/w; q^4, x^2)_\infty}{(x^2w/q^3\xi; q^4, x^2)_\infty(x^2q\xi/w; q^4, x^2)_\infty} \\
g_{x^+x^-}(w, v) &= \frac{1}{(x^2w/v; x^2)_\infty(x^2v/w; x^2)_\infty}
\end{aligned}$$

Appendix D

Normal Ordering Relations and Trace Contributions for Shiraishi's Realisation

D.1 Normal Ordering

For the normal ordering relations, we set notation

$$\mathcal{O}_1(\zeta_1)\mathcal{O}_2(\zeta_2) \;=\; \mathcal{N}_{\mathcal{O}_1\mathcal{O}_2}(\zeta_1,\zeta_2) : \mathcal{O}_1(\zeta_1)\mathcal{O}_2(\zeta_2) : \,.$$

With this, we have

$$\mathcal{N}_{\chi\Lambda_\pm}(\zeta_1,\zeta_2) \;=\; \left(\frac{(1+i\zeta_2/\zeta_1)(-q^4\zeta_2^2/\zeta_1^2;q^4)_\infty}{(-q^2\zeta_2^2/\zeta_1^2;q^4)_\infty}\right)^{\pm 1},$$

$$\mathcal{N}_{\Lambda_\pm\chi}(\zeta_1,\zeta_2) \;=\; q^{\mp\frac{1}{2}}\left(\frac{(-q^2\zeta_2^2/\zeta_1^2;q^4)_\infty}{(1+i\zeta_2/\zeta_1)(-q^4\zeta_2^2/\zeta_1^2;q^4)_\infty}\right)^{\pm 1}.$$

For the screening current $S_-(\zeta)$, we have

$$\mathcal{N}_{\chi S_-}(\zeta,z) \;=\; \zeta^{-\frac{1}{2}}\frac{(-iq^{\frac{3}{2}}z/\zeta;-q^2)_\infty}{(-iq^{\frac{1}{2}}z/\zeta;-q^2)_\infty},$$

$$\mathcal{N}_{S_-\chi}(z,\zeta) \;\; = \;\; z^{-\frac{1}{2}}\frac{(-q^{\frac{3}{2}}i\zeta/z;-q^2)_\infty}{(-q^{\frac{1}{2}}i\zeta/z;-q^2)_\infty}.$$

We also have

$$\mathcal{N}_{\chi\chi}(\zeta_1,\zeta_2) \;\; = \;\; \zeta_1^{\frac{1}{4}}\frac{(-q\zeta^{-1};-q^2)_\infty(q^6\zeta^{-2};q^4;q^4)_\infty}{(q^4\zeta^{-2};q^4;q^4)_\infty}.$$

$$\begin{aligned}
\mathcal{N}_{\Lambda_\pm S_-}(w,z) &= q^{\pm 1}\left(\frac{1-q^{-\frac{1}{2}}z/w}{1-q^{\frac{1}{2}}z/w}\right)^{\pm 1} \\
\mathcal{N}_{S_-\Lambda_\pm}(z,w) &= \frac{(1-(\zeta_2/\zeta_1)^2)}{(1-q^{-1}\zeta_2/\zeta_1)(1-q\zeta_2/\zeta_1)},
\end{aligned}$$

$$\mathcal{N}_{\Lambda_+\Lambda_-}(\zeta_1,\zeta_2) = \mathcal{N}_{\Lambda_-\Lambda_+}(\zeta_1,\zeta_2) \;\; = \;\; \frac{(1-q^{-1}\zeta_2/\zeta_1)(1-q\zeta_2/\zeta_1)}{(1-(\zeta_2/\zeta_1)^2)}.$$

D.2 Trace Contributions

Using the same notation for self contributions and pair-wise contributions as in Appendix C, we have

$$
\begin{aligned}
g_{\chi}(\zeta) &= \zeta^{\frac{1}{8}}\frac{(-qx;q^2,-q^2,x)_\infty(q^3x;q^2,-q^2,x)_\infty}{(-q^2x;q^2,-q^2,x)_\infty(q^2x;q^2,-q^2,x)_\infty}\\
g_{\chi\chi}(\zeta_1,\zeta_2) &= \frac{(-qx\zeta_1/\zeta_2;q^2,-q^2,x)_\infty(q^3x\zeta_1/\zeta_2;q^2,-q^2,x)_\infty}{(-q^2x\zeta_1/\zeta_2;q^2,-q^2,x)_\infty(q^2x\zeta_1/\zeta_2;q^2,-q^2,x)_\infty}\\
&\quad\times\frac{(-qx\zeta_2/\zeta_1;q^2,-q^2,x)_\infty(q^3x\zeta_2/\zeta_1;q^2,-q^2,x)_\infty}{(-q^2x\zeta_2/\zeta_1;q^2,-q^2,x)_\infty(q^2x\zeta_2/\zeta_1;q^2,-q^2,x)_\infty}\\
g_{S_-}(z) &= z^{\frac{1}{2}}\frac{(x;-q^2,x)_\infty(-qx;-q^2,x)_\infty}{(qx;-q^2,x)_\infty(-q^2x;-q^2,x)_\infty}\\
g_{S_-S_-}(z_1,z_2) &= \frac{(xz_1/z_2;-q^2,x)_\infty(-qxz_1/z_2;-q^2,x)_\infty}{(qxz_1/z_2;-q^2,x)_\infty(-q^2xz_1/z_2;-q^2,x)_\infty}\\
&\quad\times\frac{(xz_2/z_1;-q^2,x)_\infty(-qxz_2/z_1;-q^2,x)_\infty}{(qxz_2/z_1;-q^2,x)_\infty(-q^2xz_2/z_1;-q^2,x)_\infty}\\
g_{\Lambda_\pm} &= i^{\pm1}\frac{(x;x)_\infty}{(q^{-1}x;x)_\infty(qx;x)_\infty}\\
g_{\Lambda_\pm\Lambda_\pm}(\zeta_1,\zeta_2) &= -\frac{((\zeta_2/\zeta_1)^2x;x)_\infty}{(q^{-1}x\zeta_2/\zeta_1;x)_\infty(qx\zeta_2/\zeta_1;x)_\infty}\frac{((\zeta_1/\zeta_2)^2x;x)_\infty}{(q^{-1}x\zeta_1/\zeta_2;x)_\infty(qx\zeta_1/\zeta_2;x)_\infty}\\
g_{\Lambda_\pm\Lambda_\mp}(\zeta_1,\zeta_2) &= \frac{(q^{-1}x\zeta_2/\zeta_1;x)_\infty(qx\zeta_2/\zeta_1;x)_\infty}{((\zeta_2/\zeta_1)^2x;x)_\infty}\frac{(q^{-1}x\zeta_1/\zeta_2;x)_\infty(qx\zeta_1/\zeta_2;x)_\infty}{((\zeta_1/\zeta_2)^2x;x)_\infty}\\
g_{\chi S_+}(\zeta,z) &= (x\zeta/z;x)_\infty(xz/\zeta;x)_\infty\\
g_{\chi\Lambda_\pm}(\zeta,z) &= \left(\frac{(-iqx\zeta/z;q^2,x)_\infty(iqx\zeta/z;q^2,x)_\infty}{(-ix\zeta/z;q^2,x)_\infty(iq^2x\zeta/z;q^2,x)_\infty}\right)^{\pm1}\\
&\quad\times\left(\frac{(-iqxz/\zeta;q^2,x)_\infty(iqxz/\zeta;q^2,x)_\infty}{(-ixz/\zeta;q^2,x)_\infty(iq^2xz/\zeta;q^2,x)_\infty}\right)^{\pm1}\\
g_{\Lambda_\pm S_-}(w,z) &= \left(\frac{(xq^{\frac{1}{2}}z/w;x)_\infty}{(xq^{-\frac{1}{2}}z/w;x)\infty}\frac{(xq^{\frac{1}{2}}w/z;x)_\infty}{(xq^{-\frac{1}{2}}w/z;x)_\infty}\right)^{\pm1}\\
g_{\chi S_-}(\zeta,w) &= \frac{(-iq^{\frac{3}{2}}x\zeta/w;-q^2,x)_\infty}{(-iq^{\frac{1}{2}}x\zeta/w;-q^2,x)_\infty}\frac{(-iq^{\frac{3}{2}}xw/\zeta;-q^2,x)_\infty}{(-iq^{\frac{1}{2}}xw/\zeta;-q^2,x)_\infty}.
\end{aligned}
$$

Appendix E

Gasper-Rahman Type Integrals

Following sections 4.9 and 4.10 of [98] we will look at contour integrals of a certain form (related to the type of integrals appearing in our expressions) and compute the integrals in terms of basic hypergeometric series.

We will consider integrals of the form

$$I_m = \frac{1}{2\pi i} \oint_K P(z) z^{m-1} dz,$$

where

$$P(z) = \frac{(a_1 z, \ldots, a_A z, b_1/z, \ldots b_B/z; q)_\infty}{(c_1 z, \ldots, c_C z, d_1/z, \ldots, d_D/z; q)_\infty},$$

and the contour K is a deformation of the circle with poles of $\frac{1}{(c_1 z, \ldots, c_C z; q)_\infty}$ outside and poles of $\frac{1}{(d_1/z, \ldots, d_D/z; q)_\infty}$ inside.

The case when $D \geq B$

We work in the region $|q| < 1$ and let δ be a positive number satisfying

$$\begin{aligned} \delta &\neq |d_j q^n|, \quad ,j = 1, \dots, D, \\ \delta &\neq |c_j^{-1} q^{-n}|, \quad ,j = 1, \dots, C. \end{aligned}$$

We then define C_N as the circle $|z| = \delta |q|^N$, $C \in \mathbb{Z}_{\geq 0}$, noting that by construction, C_N does not pass through any of the poles of $P(m)$. We now consider the value of the integrand at $z = \delta q^N$:

$$\begin{aligned} & \left|P(\delta q^N)(\delta q^N)^{m-1}\right| \\ = & \left|\frac{(a_1\delta q^N, \dots, a_A\delta q^N, b_1/\delta q^N, \dots, b_B/\delta q^N; q)_\infty}{(c_1\delta q^N, \dots, c_C\delta q^N, d_1/\delta q^N, \dots, d_D/\delta; q)_\infty}\right| \cdot \left|(\delta q^N)^{m-1}\right| \\ = & \left|\frac{(a_1\delta, \dots, a_A\delta, b_1/\delta, \dots, b_B/\delta; q)_\infty}{(c_1\delta, \dots, c_C\delta, d_1/\delta, \dots, d_D/\delta q^N; q)_\infty}\right| \cdot \left|\frac{(c_1\delta, \dots, c_C\delta)_N}{(a_1\delta, \dots, a_A\delta)_N}\right| \\ & \cdot \left|\frac{(b_1/\delta q^N, \dots, b_B/\delta q^N; q)_N}{(d_1/\delta q^N, \dots d_D/\delta q^N; q)_N}\right| \cdot \left|(\delta q^N)^{m-1}\right| \\ = & \left|\frac{(a_1\delta, \dots, a_A\delta, b_1/\delta, \dots, b_B/\delta; q)_\infty}{(c_1\delta, \dots, c_C\delta, d_1/\delta, \dots, d_D/\delta; q)_\infty}\right| \cdot \left|\frac{(c_1\delta, \dots, c_C\delta)_N}{(a_1\delta, \dots, a_A\delta)_N}\right| \\ & \cdot \left|\frac{(q\delta/b_1, \dots, q\delta/b_B)_N}{(q\delta/d_1, \dots, q\delta/d_D)_N}\right| \cdot \left|\frac{b_1 \dots b_B q^{m-1}}{d_1 \dots d_D}\right|^N \cdot \left|q^{\binom{N+1}{2}}\right|^{D-B} \delta^{N(D-B)}\delta^{m-1}. \end{aligned}$$

Now, if we consider expansion in q of the above, then the term of the lowest order (i.e. of order $q^0 = 1$) gives

$$\left|P(\delta q^N)(\delta q^N)^{m-1}\right| = \mathcal{O}\left(\left|\frac{b_1 \dots b_B q^{m-1}}{d_1 \dots d_D}\right|^N \left|\delta^N q^{\binom{N+1}{2}}\right|^{D-B}\right).$$

If $D \geq B$ and

$$\left|\frac{b_1 \dots b_B q^{m-1}}{d_1 \dots d_D}\right| < 1,$$

then

$$\lim_{N\to\infty}\left(\left|\frac{b_1\dots b_B q^{m-1}}{d_1\dots d_D}\right|^N \left|\delta^N q^{\binom{N+1}{2}}\right|^{D-B}\right) = 0.$$

Since C_N is of length $\mathcal{O}\left(|q|^N\right)$, the contour shrinks arbitrarily close to the origin as we increase N. Our integrand also tends towards zero in this limit and so we have

$$\lim_{N\to\infty}\oint_{C_N} P(z)z^{m-1}dz = 0.$$

This means that we take care of any poles inside C_N, including the one at the origin. In fact, as $N\to\infty$, we assume that all other poles are now outside of C_N. Applying Cauchy's residue theorem in the region between our original contour K and C_N for large N and then letting $N\to\infty$ tells us that to compute the integral, we sum over the residues of our integrand at the poles of $\frac{1}{(d_1,\dots,d_D;q)_\infty}$. This holds for $D\geq B$ and

$$\left|\frac{b_1\dots b_B q^{m-1}}{d_1\dots d_D}\right| < 1.$$

We compute the residues using the following computation. Firstly, for $n = 0,1,2,\dots$.

$$\frac{1}{(d/z;q)_\infty} = \frac{z}{(1-d/z)(1-qd/z)\dots(1-q^{n-1}d/z)(z-q^n d)(1-q^{n+1}d/z)\dots},$$

then

$$\begin{aligned}\operatorname*{Res}_{z=dq^n} &= \frac{dq^n}{(1-1/q^n)(1-1/q^{n-1})\dots(1-q^{-1})(1-q)(1-q^2)\dots}\\ &= \frac{(-1)^n dq^n q^{\frac{n(n+1)}{2}}}{(q;q)_n(q;q)_\infty}.\end{aligned}$$

We then sum over all of our poles and obtain:

$$\frac{1}{2\pi i}\oint_K P(z)z^{m-1}dz \;=\; \sum_{i=1}^{D} f(d_i),$$

where, for example,

$$\begin{aligned} f(d_1) \;=\; & \frac{(a_1d_1,\ldots,a_Ad_1,b_1/d_1,\ldots,b_B/d_1;q)_\infty}{(q,c_1d_1,\ldots,c_Cd_1,d_2/d_1,\ldots,d_D/d_1;q)_\infty}d_1^m \\ & \cdot\sum_{n=0}^{\infty}\frac{(c_1d_1,\ldots,c_Cd_1)_n}{(q,a_1d_1,\ldots,a_Ad_1)_n}\frac{(qd_1/b_1,\ldots,qd_1/b_B)_n}{(qd_1/d_2,\ldots,qd_1/d_D)_n} \\ & \cdot\left(-d_1q^{\frac{(n+1)}{2}}\right)^{n(D-B)}\left(\frac{b_1\ldots b_Bq^m}{d_1\ldots d_D}\right)^n. \end{aligned}$$

Example E.1. *The following example shows the above process at work in a simple case where we can compute the integral by hand. We look at* $P(z)=\frac{1}{(d_1z^{-1};q)_\infty}$ *and choose* $m=0$*. Then*

$$\begin{aligned} \oint_K \frac{dz}{2\pi i}\frac{z^{-1}}{(d_1z^{-1};q)_\infty} &= \oint_K \frac{dz}{2\pi i}z^{-1}exp\left(\sum_{n>0}\frac{(d_1z^{-1})^n}{n(1-q^n)}\right) \\ &= 1, \end{aligned}$$

where we have considered the Laurent expansion of the integrand and extract the z^{-1}*th coefficient.*

Using Gasper and Rahman's approach, however, we have $D=1$*,* $A=B=C=0$ *and so, indeed,* $D>B$ *is satisfied. We now assume that* $\left|\frac{q^m}{d_1}\right|<1$ *so that our conditions are satisfied. We are supposed to now ignore the pole at* 0 *and sum over the residues of* $\frac{z^{-1}}{(d_1z^{-1};q)_\infty}$ *at the poles of* $\frac{1}{(d_1z^{-1};q)_\infty}$*. This gives*

$$\oint_K \frac{dz}{2\pi i}\frac{z^{-1}}{(d_1z^{-1};q)_\infty} \;=\; \sum_{n=0}^{\infty}\frac{(-1)^nq^{\frac{n(n+1)}{2}}}{(q;q)_n(q;q)_\infty}.$$

We now use the following formula due to Euler from Corollary 10.2.2 on page 490 of Andrews, Askey and Roy's 1999 book 'Special Functions' [99]:

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n} = (x;q)_\infty, \quad |q| < 1,$$

with the specialisation $x = q$ *and see that since* $\frac{n(n+1)}{2} = n + \binom{n}{2}$, *we have that*

$$\begin{aligned} \oint_K \frac{dz}{2\pi i} \frac{z^{-1}}{(d_1 z^{-1};q)_\infty} &= \sum_{n \geq 0} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q;q)_n (q;q)_\infty} \\ &= 1. \end{aligned}$$

This is in agreement with our previous calculation, supporting the idea that taking the limit of this special contour C_N *of such an integral deals with the pole at zero and allows us to compute the result exactly.*

The case when $C \geq A$

Of course, the specific conditions above do not always hold (and indeed do not for our purposes) and so we construct an argument parallel to Gasper and Rahman's previously demonstrated one, this time for the case when $C \geq A$ and

$$\left| \frac{a_1 \ldots a_A q^{-m+1}}{c_1 \ldots c_C} \right| < 1.$$

We now define the contour C_N as the circle $|z| = \delta^{-1}|q|^{-N}$, where δ is some positive number satisfying

$$\begin{aligned} \delta &\neq |d_j^{-1} q^{-n}|, \quad , j = 1, \ldots, D, \\ \delta &\neq |c_j q|, \quad , j = 1, \ldots, C, \end{aligned}$$

so that again C_N does not pass through the poles of the integrand. We now look at the absolute value of the integrand at $z = \delta^{-1} q^{-N}$.

$$\begin{aligned}
&\left|P(\delta^{-1}q^{-N})(\delta^{-1}q^{-N})^{m-1}\right| \\
= &\left|\frac{(a_1/\delta,\ldots,a_A/\delta,\delta b_1,\ldots,\delta b_B)_\infty}{(c_1/\delta,\ldots,c_C/\delta,\delta d_1,\ldots,\delta d_D)_\infty}\right| \cdot \left|\frac{q\delta/a_1,\ldots,q\delta/a_A)_N}{q\delta/c_1,\ldots,q\delta/c_C)_N}\right| \\
&\cdot \left|\frac{(\delta d_1,\ldots,d_D\delta;q)_N}{(\delta b_1,\ldots,b_B\delta;q)_N}\right| \cdot \left|\frac{a_1\ldots a_A q^{1-m}}{c_1\ldots c_C}\right|^N \cdot \left|q^{\binom{N+1}{2}}\right|^{C-A} \delta^{N(C-A)}\delta^{1-m}.
\end{aligned}$$

As before, we can extract the q^0th term and see that

$$\left|P(\delta^{-1}q^{-N})(\delta^{-1}q^{-N})^{m-1}\right| = \mathcal{O}\left(\left|\frac{a_1\ldots a_A q^{1-m}}{c_1\ldots c_C}\right|^N \cdot \left|\delta^N q^{\binom{N+1}{2}}\right|^{C-A}\right).$$

Then, if our conditions hold and we remember that C_N is of length $\mathcal{O}|q|^{-N}$, when we take the limit $N \to \infty$ and push our C_N contour out towards infinity, we have

$$\lim_{N\to\infty}\oint_{C_N} P(z)z^{m-1}dz = 0.$$

We can now apply Cauchy's residue theorem between K and C_N again. In doing this, we find that our integral is computed by summing the residue of the poles at $\frac{1}{(c_1z,\ldots,c_Cz,;q)_\infty}$ and is given by a formula analogous to the previous:

$$\frac{1}{2\pi i}\oint_K P(z)z^{m-1}dz = \sum_{i=0}^{C} g(c_i),$$

where e.g

$$\begin{aligned}
g(c_1) = &\frac{b_1c_1,\ldots,b_Bc_1,a_1/c_1,\ldots,a_A/c_1;q)_\infty}{(q;d_1c_1,\ldots,d_Dc_1,c_2/c_1,\ldots,c_1c_C;q)_\infty}c_1^{-m} \\
&\cdot\sum_{n=0}^{\infty}\frac{(d_1c_1,\ldots,d_Dc_1;q)_n}{(q,b_1c_1,\ldots,b_Bc_1;q)_n}\frac{(qc_1/a_1,\ldots,qc_1/a_A;q)_n}{(qc_1/c_2,\ldots,qc_1/c_C;q)_n} \\
&\cdot\left(-c_1q^{\frac{(n+1)}{2}}\right)^{n(C-A)}\left(\frac{a_1\ldots a_Aq^{-m}}{c_1\ldots c_C}\right)^n.
\end{aligned}$$

This holds when $C \geq A$ and

$$\left|\frac{a_1 \ldots a_A q^{-m+1}}{c_1 \ldots c_C}\right| < 1.$$

In the special case where we have $C \geq A$, but aso have $D = B$, we have the specialised formula:

$$\frac{1}{2\pi i}\oint_K P(z) z^{m-1} dz \quad = \quad \sum_{i=0}^{C} h(c_i),$$

where e.g.

$$\begin{aligned} h(c_1) = \quad = \quad & \frac{(b_1 c_1, \ldots, b_B c_1, a_1/c_1, \ldots, a_A/c_1; q)_\infty}{(q; d_1 c_1, \ldots, d_D c_1, c_2/c_1, \ldots, c_1 c_C; q)_\infty} c_1^{-m} \\ & \cdot {}_{A+B}\phi_{B+C-1}\left[\begin{matrix} d_1 c_1, \ldots, d_B c_1, q c_1/a_1, \ldots, q c_1/a_A \\ b_1 c_1, \ldots, b_B c_1, q c_1/c_2, \ldots, q c_1/c_C \end{matrix} ; q, u (q c_1)^{C-A}\right], \end{aligned}$$

where $u = \frac{a_1 \ldots a_A q^{-m}}{c_1 \ldots c_C}$ and ${}_r\phi_s$ is the basic hypergeometric series defined in Gasper and Rahman's notation by

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_2; q, z) \quad \equiv \quad & {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z\right] \\ = \quad & \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \end{aligned}$$

Appendix F

Explicit Integrals for the S^+ Form-Factor

We give the integrals (4.9) and (4.10) in an explicit form. Note that each expression has three terms (from each set of infinite poles) and carries over the page.

$$
\begin{aligned}
& I^R_{\Lambda_+}(\zeta_1,\zeta_2,w) \\
= \; & \frac{q^{\frac{3}{2}}}{(-q^2;-q^2)_\infty (q^2;q^2)_\infty (-q^4;-q^2,q^2)_\infty} \\
& \times \left\{ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+1} i q^{2(n+m)} q^{\frac{5}{2}}\zeta_1}{\left(\frac{1}{(-q^2)^n q^{2m}};-q^2,q^2\right)_{n,m}\left(\frac{1}{q^{2m}};q^2,-q^2\right)_{m,\infty}\left(\frac{1}{(-q^2)^n};-q^2,q^2\right)_{n,\infty}} \right. \\
& \times \frac{((-1)^n q^{1-2(n+m)};-q^2,q^2)_\infty ((-1)^{n+1} q^{2(n+m)+4};-q^2,q^2)_\infty}{((-1)^{n+1} q^{2(n+m)+3};-q^2,q^2)_\infty} \\
& \times \frac{((-1)^n q^{1-2(n+m)}\zeta_2/\zeta_1;-q^2,q^2)_\infty ((-1)^{n+1} q^{2(n+m)+4}\zeta_1/\zeta_2;-q^2,q^2)_\infty}{((-1)^n \zeta_2/q^{2(n+m)+2}\zeta_1;-q^2,q^2)_\infty ((-1)^{n+1} q^{2(n+m)+3}\zeta_1/\zeta_2;-q^2,q^2)_\infty} \\
& \times \frac{((-1)^{n+1}\zeta_2/q^{2(n+m)+2}\zeta_1;-q^2)_\infty ((-1)^{(n+1)} q^{2(n+m)+4}\zeta_1/\zeta_2;-q^2)_\infty}{((-1)^n q^{2(n+m)+4}\zeta_1/\zeta_2;-q^2)_\infty} \\
& \times \left(\frac{1+(-1)^n q^{2(n+m)} q^{\frac{5}{2}}\zeta_1/w}{1+(-1)^n q^{2(n+m)} q^{\frac{7}{2}}\zeta_1/w}\right) \Theta_{q^8}\left(\frac{q^{4(n+m)} q^5 \zeta_1}{\zeta_2}\right) \\
& \times \frac{((-1)^{n+1} q^{2(n+m)} q^{\frac{11}{2}}\zeta_1/w;q^2,q^2)_\infty}{((-1)^{n+1} q^{\frac{9}{2}} q^{2(n+m)}\zeta_1/w;q^2,q^2)_\infty} \frac{((-1)^{n+1} w/q^{\frac{1}{2}} q^{2(n+m)}\zeta_1;q^2,q^2)_\infty}{((-1)^{n+1} w/q^{2(n+m)} q^{\frac{3}{2}}\zeta_1;q^2,q^2)_\infty} \\
& \left. -iq^{\frac{5}{2}}\zeta_1 + \frac{(-1)^{n+1} i q^{\frac{5}{2}} q^{2n}\zeta_1}{\left(\frac{1}{(-q^2)^n};-q^2,q^2\right)_{n,\infty}} - \frac{i q^{\frac{5}{2}} q^{2m}\zeta_1}{\left(\frac{1}{q^{2m}};q^2,-q^2\right)_{m,\infty}} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{q^{\frac{3}{2}}}{(-q^2;-q^2)_\infty(q^2;q^2)_\infty(-q^4;-q^2,q^2)_\infty}\\
&\times\Bigg\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}iq^{2(n+m)}q^{\frac{1}{2}}\zeta_2}{\left(\frac{1}{(-q^2)^nq^{2m}};-q^2,q^2\right)_{n,m}\left(\frac{1}{q^{2m}};q^2,-q^2\right)_{m,\infty}\left(\frac{1}{(-q^2)^n};-q^2,q^2\right)_{n,\infty}}\\
&\times\frac{((-1)^nq^{3-2(n+m)}\zeta_1/\zeta_2;-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+2}\zeta_2/\zeta_1;-q^2,q^2)_\infty}{((-1)^nq^{2(n+m)+1}\zeta_1/\zeta_2;-q^2,q^2)_\infty}\\
&\times\frac{((-1)^nq^3/q^{2(n+m)};-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+2};-q^2,q^2)_\infty}{((-1)^n/q^{2(n+m)};-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+1};-q^2,q^2)_\infty}\\
&\times\frac{((-1)^{n+1}/q^{2(n+m)};-q^2)_\infty((-1)^nq^{2(n+m)+2};-q^2)_\infty}{((-1)^{n+1}q^{2(n+m)+2};-q^2)_\infty}\\
&\times\left(\frac{1+(-1)^nq^{2(n+m)}q^{\frac{1}{2}}\zeta_2/w}{1+(-1)^nq^{2(n+m)}q^{\frac{3}{2}}\zeta_2/w}\right)\Theta_{q^8}\left(\frac{q^{4(n+m)}q}{\zeta_1}\right)\\
&\times\frac{((-1)^{n+1}q^{2(n+m)}q^{\frac{7}{2}}\zeta_2/w;q^2,q^2)_\infty}{((-1)^{n+1}q^{2(n+m)}q^{\frac{5}{2}}\zeta_2/w;q^2,q^2)_\infty}\frac{((-1)^{n+1}q^{\frac{3}{2}}w/q^{2(m+n)}\zeta_2;q^2,q^2)_\infty}{((-1)^{n+1}q^{\frac{1}{2}}w/q^{2(m+n)}/\zeta_2;q^2;q^2)_\infty}\\
&-iq^{\frac{5}{2}}\zeta_1+\frac{(-1)^{n+1}iq^{\frac{1}{2}}q^{2n}\zeta_2}{\left(\frac{1}{(-q^2)^n};-q^2,q^2\right)_{n,\infty}}-\frac{iq^{\frac{1}{2}}q^{2m}\zeta_2}{\left(\frac{1}{q^{2m}};q^2,-q^2\right)_{m,\infty}}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{q^{\frac{3}{2}}}{(q^2;q^2)_\infty(q^2;q^2)_\infty(q^4;q^2,q^2)_\infty}\\
&\times\Bigg\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{iwq^{2(n+m)+1}}{(1/(q^{2(n+m)};q^2,q^2)_{n,m}(1/q^{2m};q^2,q^2)_{m,\infty}(1/q^{2n};q^2,q^2)_{n,\infty}}\\
&\times\frac{(-q^{\frac{9}{2}}\zeta_1/q^{2(n+m)}w;-q^2,q^2)_\infty(q^{\frac{5}{2}}q^{2(n+m)}w/\zeta_1;-q^2,q^2)_\infty}{(-q^{\frac{3}{2}}\zeta_1/q^{2(n+m)}w;-q^2,q^2)_\infty(q^{\frac{3}{2}}q^{(2(n+m)}w/\zeta_1;-q^2,q^2)_\infty}\\
&\times\frac{(-q^{\frac{5}{2}}\zeta_2/q^{(2(n+m)}w;-q^2,q^2)_\infty(q^{\frac{5}{2}}q^{(2(n+m)}w/\zeta_2;-q^2,q^2)_\infty}{(-\zeta_2/q^{\frac{1}{2}}q^{(2(n+m)}w;-q^2,q^2)_\infty(q^{\frac{3}{2}}q^{2(n+m)}w/\zeta_2;-q^2,q^2)_\infty}\\
&\times\frac{(\zeta_2/q^{\frac{1}{2}}q^{2(n+m)}w;-q^2)_\infty(-q^{\frac{5}{2}}q^{(2(n+m)}w/\zeta_2;-q^2)_\infty}{(q^{\frac{5}{2}}q^{(2(n+m)}w/\zeta_2;-q^2)_\infty}\left(\frac{1-q^{2(n+m)+1}}{1-q^{2(n+m)+2}}\right)\\
&\times\Theta_{q^8}\left(\frac{q^2w^2q^{4(n+m)}}{\zeta_1\zeta_2}\right)\frac{q^{(2(n+m)+4};q^2,q^2)_\infty}{(q^{(2(n+m)+3};q^2,q^2)_\infty}(q^2/q^{(2(n+m)};q^2,q^2)_\infty\\
&+iqw+\frac{iwq^{2n+1}\zeta_2}{(1/q^{2n};q^2,q^2)_{n,\infty}}+\frac{iwq^{2m+1}\zeta_2}{(1/q^{2m};q^2,q^2)_{m,\infty}}\Bigg\}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
& I^R_{\Lambda_-}(\zeta_1,\zeta_2,w) \\
= \quad & \frac{q^{-\frac{3}{2}}}{(-q^2;-q^2)_\infty(q^2;q^2)_\infty(-q^4;-q^2,q^2)_\infty} \\
& \times\Bigg\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}iq^{2(n+m)}q^{\frac{5}{2}}\zeta_1}{\left(\frac{1}{(-q^2)^n q^{2m}};-q^2,q^2\right)_{n,m}\left(\frac{1}{q^{2m}};q^2,-q^2\right)_{m,\infty}\left(\frac{1}{(-q^2)^n};-q^2,q^2\right)_{n,\infty}} \\
& \times\frac{((-1)^n q^{1-2(n+m)};-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+4};-q^2,q^2)_\infty}{((-1)^{n+1}q^{2(n+m)+3};-q^2,q^2)_\infty} \\
& \times\frac{((-1)^n q^{1-2(n+m)}\zeta_2/\zeta_1;-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+4}\zeta_1/\zeta_2;-q^2,q^2)_\infty}{((-1)^n\zeta_2/q^{2(n+m)+2}\zeta_1;-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+3}\zeta_1/\zeta_2;-q^2,q^2)_\infty} \\
& \times\frac{((-1)^{n+1}\zeta_2/q^{2(n+m)+2}\zeta_1;-q^2)_\infty((-1)^{(n+1)}q^{2(n+m)+4}\zeta_1/\zeta_2;-q^2)_\infty}{((-1)^n q^{2(n+m)+4}\zeta_1/\zeta_2;-q^2)_\infty} \\
& \times\left(\frac{1-(-1)^n q^{2(n+m)}q^{\frac{5}{2}}\zeta_1/w}{1-(-1)^n q^{2(n+m)}q^{\frac{3}{2}}\zeta_1/w}\right)\Theta_{q^8}\left(\frac{q^{4(n+m)}q^5\zeta_1}{\zeta_2}\right) \\
& \times\frac{((-1)^n q^{2(n+m)}q^{\frac{7}{2}}\zeta_1/w;q^2,q^2)_\infty}{((-1)^n q^{\frac{9}{2}}q^{2(n+m)}\zeta_1/w;q^2,q^2)_\infty}\frac{((-1)^n w/q^{\frac{1}{2}}q^{2(n+m)}\zeta_1;q^2,q^2)_\infty}{((-1)^n q^{\frac{1}{2}}w/q^{2(n+m)}\zeta_1;q^2,q^2)_\infty} \\
& -iq^{\frac{5}{2}}\zeta_1+\frac{(-1)^{n+1}iq^{\frac{5}{2}}q^{2n}\zeta_1}{\left(\frac{1}{(-q^2)^n};-q^2,q^2\right)_{n,\infty}}-\frac{iq^{\frac{5}{2}}q^{2m}\zeta_1}{\left(\frac{1}{q^{2m}};q^2,-q^2\right)_{m,\infty}}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{q^{-\frac{3}{2}}}{(-q^2;-q^2)_\infty(q^2;q^2)_\infty(-q^4;-q^2,q^2)_\infty}\\
&\times\Bigg\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}iq^{2(n+m)}q^{\frac{1}{2}}\zeta_2}{\left(\frac{1}{(-q^2)^nq^{2m}};-q^2,q^2\right)_{n,m}\left(\frac{1}{q^{2m}};q^2,-q^2\right)_{m,\infty}\left(\frac{1}{(-q^2)^n};-q^2,q^2\right)_{n,\infty}}\\
&\times\frac{((-1)^nq^{3-2(n+m)}\zeta_1/\zeta_2;-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+2}\zeta_2/\zeta_1;-q^2,q^2)_\infty}{((-1)^nq^{2(n+m)+1}\zeta_1/\zeta_2;-q^2,q^2)_\infty}\\
&\times\frac{((-1)^nq^3/q^{2(n+m)};-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+2};-q^2,q^2)_\infty}{((-1)^n/q^{2(n+m)};-q^2,q^2)_\infty((-1)^{n+1}q^{2(n+m)+1};-q^2,q^2)_\infty}\\
&\times\frac{((-1)^{n+1}/q^{2(n+m)};-q^2)_\infty((-1)^nq^{2(n+m)+2};-q^2)_\infty}{((-1)^{n+1}q^{2(n+m)+2};-q^2)_\infty}\\
&\times\left(\frac{1-(-1)^nq^{2(n+m)}q^{\frac{1}{2}}\zeta_2/w}{1-(-1)^nq^{2(n+m)}q^{-\frac{1}{2}}\zeta_2/w}\right)\Theta_{q^8}\left(\frac{q^{4(n+m)}q}{\zeta_1}\right)\\
&\times\frac{((-1)^nq^{2(n+m)}q^{\frac{3}{2}}\zeta_2/w;q^2,q^2)_\infty}{((-1)^nq^{2(n+m)}q^{\frac{5}{2}}\zeta_2/w;q^2,q^2)_\infty}\frac{((-1)^nq^{\frac{3}{2}}w/q^{2(m+n)}\zeta_2;q^2,q^2)_\infty}{((-1)^nq^{\frac{5}{2}}w/q^{2(m+n)}\zeta_2;q^2;q^2)_\infty}\\
&-iq^{\frac{5}{2}}\zeta_1+\frac{(-1)^{n+1}iq^{\frac{1}{2}}q^{2n}\zeta_2}{\left(\frac{1}{(-q^2)^n};-q^2,q^2\right)_{n,\infty}}-\frac{iq^{\frac{1}{2}}q^{2m}\zeta_2}{\left(\frac{1}{q^{2m}};q^2,-q^2\right)_{m,\infty}}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{q^{-\frac{3}{2}}}{(q^2;q^2)_\infty(q^2;q^2)_\infty(q^4;q^2,q^2)_\infty}\\
&\times\Bigg\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{-iwq^{2(n+m)+3}}{(1/(q^{2(n+m)};q^2,q^2)_{n,m}(1/q^{2m};q^2,q^2)_{m,\infty}(1/q^{2n};q^2,q^2)_{n,\infty}}\\
&\times\frac{(q^{\frac{1}{2}}\zeta_1/wq^{2(n+m)};-q^2,q^2)_\infty(-q^{\frac{9}{2}}wq^{2(n+m)}/\zeta_1;-q^2,q^2)_\infty}{(q^{-\frac{1}{2}}\zeta_1/wq^{2(n+m)};-q^2,q^2)_\infty(-q^{\frac{7}{2}}wq^{2(n+m)}/\zeta_1;-q^2,q^2)_\infty}\\
&\times\frac{(q^{\frac{1}{2}}\zeta_2/wq^{2(n+m)};-q^2,q^2)_\infty(-q^{\frac{9}{2}}q^{2(n+m)}w/\zeta_2;-q^2,q^2)_\infty}{(q^{-\frac{5}{2}}\zeta_2/wq^{2(n+m)};-q^2,q^2)_\infty(q^{\frac{7}{2}}wq^{2(n+m)}/\zeta_2;-q^2,q^2)_\infty}\\
&\times\frac{(-q^{-\frac{5}{2}}\zeta_2/wq^{2(n+m)};-q^2)_\infty(q^{\frac{9}{2}}wq^{2(n+m)}/\zeta_2;-q^2)_\infty}{(-q^{\frac{9}{2}}wq^{2(n+m)}/\zeta_2;-q^2)_\infty}\left(\frac{1-q^{2(n+m)+3}}{1-q^{2(n+m)+2}}\right)\\
&\times\Theta_{q^8}\left(-\frac{z^2}{\zeta_1\zeta_2}\right)\frac{(q^{2(n+m)+4};q^2,q^2)_\infty}{(q^{2(n+m)+5};q^2,q^2)_\infty}(1/q^{2(n+m)+1};q^2,q^2)_\infty
\end{aligned}
$$

Appendix G

Normal Ordering Relations for q-Wakimoto Formalism

This appendix lists all operator product expansion formulae required in the q-Wakimoto formalism. We consider the normal ordering of Drinfeld currents $J^+(u)$ and $J^-(w)$, screening current $S(t)$ and type I and type II spin $\frac{l}{2}$, level k naïve vertex operators $\phi_{l,l}(z)$ and $\psi_{l,0}(z)$, respectively. The free field realisation of this collection of operators is given below.

$$
\begin{aligned}
\phi_{l,l}(z) &= \, : \exp\left\{ a\left(l;\, 2, k+2 \,|\, q^k z;\, \frac{k+2}{2} \right) \right\} : , \\
\psi_{l,0}(z) &= \, : \exp\Bigg\{ a\left(l; 2, k+2 \,|\, q^{k-2} z; -\frac{k+2}{2} \right) \\
&\qquad\qquad + b\left(l; 2, 1 \,|\, q^{-2} z; 0 \right) + c\left(l; 2, 1 \,|\, q^{-2} z; 0 \right) \Bigg\} : , \\
S(t) &= -\frac{1}{(q-q^{-1})t} \left(S_1(t) - S_{-1}(t) \right), \\
J^+(u) &= \frac{1}{(q-q^{-1})u} \left(J^+_1(u) - J^+_{-1}(u) \right) \\
J^-(w) &= \frac{1}{(q-q^{-1})w} \left(J^-_1(w) - J^-_{-1}(w) \right),
\end{aligned}
$$

where

$$
\begin{aligned}
S_1(t) &= :\exp\Bigg\{-a\left(k+2|q^{-2}t;-\frac{k+2}{2}\right) \\
&\quad -b\left(2|q^{-k-2}t;-1\right)-c\left(2|q^{-k-1}t;0\right)\Bigg\}, \\
S_{-1}(t) &= :\exp\Bigg\{-a\left(k+2|q^{-2}t;-\frac{k+2}{2}\right) \\
&\quad -b\left(2|q^{-k-2}t;-1\right)-c\left(2|q^{-k-3}t;0\right)\Bigg\}, \\
J_1^+(u) &= :\exp\left\{-b\left(2|q^{-k-2}u;1\right)-c\left(2|q^{-k-1}u;0\right)\right\}:, \\
J_{-1}^+(u) &= :\exp\left\{-b\left(2|q^{-k-2}u;1\right)-c\left(2|q^{-k-3}u;0\right)\right\}:, \\
J_1^-(w) &= :\exp\Bigg\{a\left(k+2|q^{k}w;-\frac{k+2}{2}\right)-a\left(k+2|q^{-2}w;\frac{k+2}{2}\right) \\
&\quad +b\left(2|w;-1\right)+c\left(2|q^{-1}w;0\right)\Bigg\}:, \\
J_{-1}^-(w) &= :\exp\Bigg\{a\left(k+2|q^{-k-4}w;-\frac{k+2}{2}\right)-a\left(k+2|q^{-2}w;\frac{k+2}{2}\right) \\
&\quad +b\left(2|q^{-2k-4}w;-1\right)+c\left(2|q^{-2k-3}w;0\right)\Bigg\}:.
\end{aligned}
$$

G.1 $U_q(\widehat{sl_2})$ Currents and Screening Operators

$$
\begin{aligned}
S_1(t_1)S_1(t_2) &= t_1^{\frac{2}{k+2}}\frac{(q^{-2}pt_2/t_1;p)_\infty}{(q^2t_2/t_1;p)_\infty}q(1-t_2/t_1):S_1(t_1)S_1(t_2): \\
S_1(t_1)S_{-1}(t_2) &= -t_1^{\frac{2}{k+2}}\frac{(q^{-2}pt_2/t_1;p)_\infty}{(q^2t_2/t_1;p)_\infty}q(1-q^{-2}t_2/t_1):S_1(t_1)S_{-1}(t_2): \\
S_{-1}(t_1)S_1(t_2) &= -t_1^{\frac{2}{k+2}}\frac{(q^{-2}pt_2/t_1;p)_\infty}{(q^2t_2/t_1;p)_\infty}q^{-1}(1-q^2t_2/t_1):S_{-1}(t_1)S_1(t_2): \\
S_{-1}(t_1)S_{-1}(t_2) &= t_1^{\frac{2}{k+2}}\frac{(q^{-2}pt_2/t_1;p)_\infty}{(q^2t_2/t_1;p)_\infty}q^{-1}(1-t_2/t_1):S_{-1}(t_1)S_{-1}(t_2):
\end{aligned}
$$

$$\begin{aligned}
S_1(t)J_1^+(u) &= q : S_1(t)J_1^+(u) :, \\
S_1(t)J_{-1}^+(u) &= q\frac{t-q^{-2}u}{t-u} : S_1(t)J_{-1}^+(u) :, \quad |t| > q^{-2}|u|, \\
S_{-1}(t)J_1^+(u) &= q^{-1}\frac{t-q^{2}u}{t-u} : S_{-1}(t)J_1^+(u) :, |t| > |u|, \\
S_{-1}(t)J_{-1}^+(u) &= q^{-1} : S_{-1}(t)J_{-1}^+(u) :, \\
J_1^+(u)S_1(t) &= q : S_1(t)J_1^+(u) :, \\
J_1^+(u)S_{-1}(t) &= q\frac{u-q^{-2}t}{u-t} : S_{-1}(t)J_1^+(u) :, |u| > q^{-2}|t|, \\
J_{-1}^+(u)S_1(t) &= q^{-1}\frac{u-q^{2}t}{u-t} : S_1(t)J_{-1}^+(u) :, |u| > |t|, \\
J_{-1}^+(u)S_{-1}(t) &= q^{-1} : S_{-1}(t)J_{-1}^+(u) : .
\end{aligned}$$

$$\begin{aligned}
S_1(t)J_1^-(w) &= q^{-1} : S_1(t)J_1^-(w) :, \\
S_1(t)J_{-1}^-(w) &= q^{-1}\frac{t-q^{-k}w}{t-q^{-k-2}w} : S_1(t)J_{-1}^-(w) :, \quad |t| > q^{-k-2}|w|, \\
S_{-1}(t)J_1^-(w) &= q\frac{t-q^{k}w}{t-q^{k+2}w} : S_{-1}(t)J_1^-(w) :, \quad |t| > q^{k}|w|, \\
S_{-1}(t)J_{-1}^-(w) &= q : S_{-1}(t)J_{-1}^-(w) :, \\
J_1^-(w)S_1(t) &= q^{-1} : S_1(t)J_1^-(w) :, \\
J_1^-(w)S_{-1}(t) &= q^{-1}\frac{w-q^{-k}t}{w-q^{-k-2}t} : S_{-1}(t)J_1^-(w) :, \quad |w| > q^{-k-2}|t|, \\
J_{-1}^-(w)S_1(t) &= q\frac{w-q^{k}t}{w-q^{k+2}t} : S_1(t)J_{-1}^-(w) :, \quad |w| > q^{k}|t|, \\
J_{-1}^-(w)S_{-1}(t) &= q : S_{-1}(t)J_{-1}^-(w) : .
\end{aligned}$$

G.2 Spin $\frac{l}{2}$, Level k, Type I

Normal ordering two of these vertex operators gives us:

$$\begin{aligned}
\phi_{l,l}(z_1)\phi_{l,l}(z_2) &= (q^k z_1)^{\frac{l^2}{2(k+2)}} \frac{(pq^{2(1-l)}z_2/z_1;q^4,p)_\infty (pq^{2(1+l)}z_2/z1;q^4,p)_\infty}{(pq^2 z_2/z_1;q^4,p)_\infty^2} \\
&\quad \times : \phi_{l,l}(z_1)\phi_{l,l}(z_2) :, \\
&\quad |z_1| > q^{2(1-l)}|z_2|.
\end{aligned}$$

For the Drinfeld currents, we have the following:

$$\begin{aligned}
J_1^-(w)\phi_{l,l}(z) &= \frac{q^l w - q^{k+2}}{w - q^{l+k+2}z} : J_1^-(w)\phi_{l,l}(z) : , \quad |w| > q^{k+2-l}|z|, \\
J_{-1}^-(w)\phi_{l,l}(z) &= q^{-l} : J_{-1}^-(w)\phi_{l,l}(z) : , \\
\phi_{l,l}(z)J_1^-(w) &= : J_1^-(w)\phi_{l,l}(z) : , \\
\phi_{l,l}(z)J_{-1}^-(w) &= \frac{z - q^{-l-k-2}w}{z - q^{l-k-2}w} : J_{-1}^-(w)\phi_{l,l}(z) : , \quad |z| > q^{-l-k-2}|w|.
\end{aligned}$$

Noting a correction to the relations appearing in [64], we have the following relations between the type I vertex operator and the screening operator $S(t)$.

$$\begin{aligned}
S(t)\phi_{l,l}(z) &= (q^{-2}t)^{-\frac{l}{k+2}} \frac{(q^l pz/t; p)_\infty}{(q^{-l}pz/t; p)_\infty} : S(t)\phi_{2,2}(z) : , \quad |t| > q^{-l}p|z|, \\
\phi_{l,l}(z)S(t) &= (q^l z)^{-\frac{2}{k+2}} \frac{(q^2 t/z; p)_\infty}{(q^{-2}t/z; p)_\infty} : S(t)\phi_{2,2}(z) : , \quad |z| > q^{-l}|t|.
\end{aligned}$$

G.3 Spin $\frac{l}{2}$, Level k, Type II

For two type II vertex operators, we have

$$\psi_{l,0}(z_1)\psi_{l,0}(z_2) = (q^{k-2}z_1)^{\frac{l^2}{2(k+2)}} \frac{(q^{2(l-1)}z_2/z_1; q^4, p)_\infty}{(q^{2(l+1)}z_2/z_1; q^4, p)_\infty} (q^2 z_2/z_1; q^4, p)_\infty^2 : \psi_{l,0}(z_1)\psi_{l,0}(z_2) : .$$

This looks very like the relation for $\phi_{l,l}(z_1)\phi_{l,l}(z_2)$ because the contributions to the zero modes from the a field are the same. We now normal order $\psi_{l,0}(z)$ with the Drinfeld currents as follows.

$$\begin{aligned}
J_1^+(u)\psi_{l,0}(z) &= q^{-l}\frac{u - q^{k+l}z}{u - q^{k-l}z} : J_1^+(u)\psi_{l,0}(z) : , \quad |u| > q^{k-l}|z|, \\
J_{-1}^+(u)\psi_{l,0}(z) &= q^l : J_{-1}^+(u)\psi_{l,0}(z) : , \\
\psi_{l,0}(z)J_1^+(u) &= : J_1^+(u)\psi_{l,0}(z) : , \\
\psi_{l,0}(z)J_{-1}^+(u) &= \frac{z - q^{l-k}u}{z - q^{-l-k}u} : J_{-1}^+(u)\psi_{l,0}(z) : , \quad |z| > q^{-l-k}|u|.
\end{aligned}$$

Finally, for the screening current, we have

$$
\begin{aligned}
S_1(t)\psi_{1,0}(z) &= (q^{-2}t)^{-\frac{1}{k+2}}q^{-1}\frac{(q^{k+1}z/t;p)_\infty}{(q^{k-1}z/t;p_\infty} : S_1(t)\psi_{1,0}(z) :, \quad |t| > q^{k-1}|z|, \\
S_{-1}(t)\psi_{1,0}(z) &= (q^{-2}t)^{-\frac{1}{k+2}}q^{-1}\frac{(q^{k+1}pz/t;p)_\infty}{(q^{k-1}pz/t;p_\infty} : S_{-1}(t)\psi_{1,0}(z) :, \quad |t| > q^{k-1}p|z|, \\
\psi_{1,0}(z)S_1(t) &= (q^{k-2}z)^{-\frac{1}{k+2}}q^{-1}\frac{(q^{k+1}pt/z;p)_\infty}{(q^{k-1}pt/z;p_\infty} : S_1(t)\psi_{1,0}(z) :, \quad |z| > q^{k-1}p|z|, \\
\psi_{1,0}(z)S_{-1}(t) &= (q^{k-2}z)^{-\frac{1}{k+2}}q^{-1}\frac{(q^{k+1}t/z;p)_\infty}{(q^{k-1}t/z;p_\infty} : S_{-1}(t)\psi_{1,0}(z) :, \quad |z| > q^{k-1}|t|.
\end{aligned}
$$

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